SOME GEOMETRIC PROPERTIES OF THE SPHERES IN A NORMED LINEAR SPACE

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1. **Introduction.** Recently two different generalizations of Clarkson's uniform convexity [2] have been studied. In [8], Lovaglia introduced locally uniformly convex spaces. According to him, a normed linear space X is *locally uniformly convex*, if, for any $x_0 \in X$ with $\| x_0 \| = 1$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that $\| (x_0 + x)/2 \| \leq 1 - \delta$ whenever $\| x \| = 1$ and $\| x - x_0 \| \geq \epsilon$. In [4], the authors of the present paper have introduced fully k-convex spaces. For any fixed integer $k \geq 2$, a normed linear space X is called *fully k-convex*, if every sequence $\{x_n\}$ of elements in X satisfying $\lim_{n \to \infty} \| x_n \| = 1$ and $\| 1/k \sum_{i=1}^{k} x_{\nu_i} \| \geq 1$ for any k indices $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_k$ is a Cauchy sequence. It is easily seen that if a normed linear space is uniformly convex, then it is locally uniformly convex and also fully k-convex for any $k \geq 2$. The converse of this is known to be false (see [8], [4]).

In the present paper, we shall study several similar convexity properties of the spheres in a normed linear space. These properties are listed in §2 and labeled (A)–(H); those which we know to be equivalent in any normed linear space are labeled with the same letter, followed by a distinguishing numeral ((e.g. (C.1), (C.2)).

Among the properties considered, (B), (C.1) and (D) clearly generalize the defining condition of full k-convexity, while condition (F) generalizes local uniform convexity. Condition (E.2) has been previously studied by Šmulian [10], [11]. Also, (E.3) is a well-known property of uniformly convex spaces (see [9; 402]), and (H) is a well-known and useful property of Hilbert space (see [1; 245]).

In §3 we investigate the relationship between the properties in a normed linear space, showing, in Theorem 2, that (with one exception) each implies its successor. Then in §4, this comparison is made in a Banach space. It turns out (Theorem 3) that for a Banach space, conditions (A), (B), (C), (D), (E) are mutually equivalent, and are also equivalent to each of (F), (G), (H) combined with reflexivity. From Theorem 2 and the equivalent formulations of full k-convexity given in [4, Theorem 1], it is easily seen that, for any $k \geq 2$, every fully k-convex normed linear space possesses the properties (C), (D), (E), (G) and (H) (if $k = 2$, (B) also holds). From Theorems 2 and 3, every locally uniformly convex normed linear space has properties (F), (G), (H), and every fully k-convex Banach space or reflexive locally uniformly convex Banach space has all the properties (A)–(H).

In §5 and §6 we study some further connections of our properties with uniform convexity or local uniform convexity. Theorem 4 gives two new character-

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