INEQUALITIES FOR NORMAL AND HERMITIAN MATRICES

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1. Throughout this note A will stand for a complex $n \times n$ matrix with characteristic roots $\omega_1, \cdots, \omega_n$. The *spread* $s(A)$ of A is defined by the equation

$$s(A) = \max_{r,s} | \omega_r - \omega_s |.$$

In an earlier paper [5] (which will be referred to as S) I obtained an upper bound for $s(A)$, valid for every matrix A, and lower bounds for the case of Hermitian and normal matrices. This last case is now to be pursued a little further. In the first place it will be shown that the lower estimates for $s(A)$, obtained previously, are best possible. Then a number of further estimates will be derived, and $s(A)$ will be characterized by various maximum properties. Finally, a function of the characteristic roots of A, more general than $s(A)$, will be considered.

We shall denote by a_{rs} the (r, s)-th element of A, and by e_k the column vector whose k-th component is 1 while all other components are 0. Transposed conjugates will, as usual, be indicated by an asterisk.

2. We begin with a number of results relating to Hermitian matrices.

Theorem 1. *If A is a Hermitian matrix, then*

$$s(A) = 2 \sup_{u,v} | u^*Av |,$$

where the upper bound is taken with respect to all pairs of orthonormal vectors u, v.

Assume, without loss of generality, that $s(A) = \omega_1 - \omega_2$. Let T be a unitary matrix such that

$$T^*AT = \operatorname{diag}(\omega_1, \omega_2, \cdots, \omega_n).$$

If the first and the second columns of T are denoted by x and y respectively, then $\| x \| = \| y \| = 1$, $x^*y = 0$; and $x^*Ax = \omega_1$, $y^*Ay = \omega_2$. Hence

$$s(A) = x^*Ax - y^*Ay. \tag{1}$$

Now put

$$u_0 = (x + iy)/\sqrt{2}, \qquad v_0 = (x - iy)/\sqrt{2}, \qquad p = x^*Ay.$$

Then $\| u_0 \| = \| v_0 \| = 1$, $u_0^*v_0 = 0$; and using (1) and the fact that A is Hermitian, we obtain

$$u_0^*Av_0 = \tfrac{1}{2}\{s(A) - 2i\Re p\}.$$

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