# NON-LINEAR FUNCTIONAL EQUATIONS IN LOCALLY CONVEX SPACES

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In various problems of non-linear analysis, topological methods may be applied to show the existence and determine the number of solutions of non-linear functional equations defined in function spaces. The primary purpose of the present paper is to give precise and simple proofs of uniqueness and multiplicity theorems for solutions of functional equations in the form in which they may be applied to the determination of the actual number of solutions of a boundary value problem like the Dirichlet problem for a non-linear elliptic partial differential equation. The discussion is divided into two parts, the first comprising §§1 and 2, applying the theory of the Leray-Schauder degree, the second contained in §3 being based entirely upon the general topological theory of covering spaces.

The methods which are applied here are based upon rather classical ideas. The theory of the degree stems from the theory within combinatorial topology of the fixed points and the degree of continuous mappings as originated by Brouwer (see [1] for expository and historical discussion) and applied in function spaces to analytical problems by Birkhoff and Kellogg [3], Schauder [22], [23], [24], Leray and Schauder [15], and Leray [11], [12], [14]. The application of the theory of covering mappings stems from the monodromy argument of classical function theory, whose ideas were first applied to the study of solutions of functional equations by Hadamard [9] and P. Levy [16]. Yet, though the ideas are classical, a systematic development of their application to the uniqueness and multiplicity problem seems never to have been given. In particular, although Cacciopoli [6] stated results in Banach space using the "continuity" or "monodromy" argument, and applied these results to the uniqueness of the solutions of elliptic equations, one searches in vain through the available mathematical literature for a complete and rigorous proof under precise hypotheses of the most general results in this direction. Precise statements of such results in Banach spaces have been given by Miranda in [17; 140–145] without proof. Since interesting applications to elliptic equations can be given which go beyond this setting, it has seemed of value to the writer to present a theory of this type in the most general setting. In addition to providing an easily accessible proof of general theorems of this type, such a systematic development may also remove some of the aura of doubt that seems to envelop, for some, the applications of the "continuity" method in elliptic partial differential equations. We emphasize that the results stated below involve no assumptions of the existence or uniform invertibility of the Frechet derivative of the mapping concerned,

Received February 8, 1957; in revised form, May 16, 1957.