A NOTE CONCERNING REGULAR MEASURES

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1. **Introduction.** Let X be a topological space, and let $(X, \mathbf{S}, \mu)$ be an associated measure space. We shall say that a measurable set E is *inner regular with respect to* μ if

$$\mu(E) = \sup \{\mu(C): E \supset C, C \in \mathbf{C}\},$$

where $\mathbf{C}$ is the class of compact measurable sets. The measurable set E will be called *outer regular with respect to* μ if

$$\mu(E) = \inf \{\mu(U): E \subset U, U \in \mathbf{U}\},$$

where $\mathbf{U}$ is the class of open measurable sets. If each measurable set is inner (outer) regular, the measure μ will be termed *inner* (*outer*) *regular*.

Let $f(x)$ be any non-negative measurable ($\mathbf{S}$) function on X. Define the measure ν on $\mathbf{S}$ by means of the equation

$$\nu(E) = \int_E f(x)\, d\mu(x).$$

Then, if μ is characterized by some property of regularity, it is natural to inquire whether ν is similarly distinguished. It is the purpose of this note to give a brief exposition of the conditions under which the regularity properties of μ will be induced on ν.

A result in this general direction has been obtained by Hahn and Rosenthal [1; 175]. Let φ be a countably additive (not necessarily non-negative) set function on $\mathbf{S}$, and let $\bar{\varphi}$ be its absolute function. (The terms *signed measure* and *total variation* are also used for φ and $\bar{\varphi}$ respectively [2].) The set function φ is said to be *content-like* if there exists for every measurable set E, a measurable G_δ set, F, which contains E and for which $\bar{\varphi}(F) = \bar{\varphi}(E)$.

Theorem. *If φ is countably additive, σ-finite and content-like and if $f(x)$ is φ-integrable (in the sense of Hahn and Rosenthal) then the φ-integral of f is also content-like.*

Although it is clear that the concepts of outer regularity and content-likeness are very closely related to one another, we shall see that the analogue of this theorem with outer regular in place of content-like is not necessarily true.

2. **Inner regularity.** Our first result is an affirmative answer to the question of the introduction, in the case of inner regular measures.

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