A THEOREM OF HAMILTON: COUNTEREXAMPLE

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The following appears in a paper of O. H. Hamilton [1].

THEOREM. *If I_n is a topological n-cell, T is a continuous multi-valued transformation of I_n into a subset of itself such that for each point P of I_n, $T(P)$ is the boundary of a topological n-cell and M_1, M_2, and M_3 are the subsets of I_n consisting of the points P which are respectively in the interior of $T(P)$, in $T(P)$, and in the exterior of $T(P)$, then* (a) *M_2 is non-vacuous and closed,* (b) *$M_1 + M_2$ and $M_2 + M_3$ are each closed,* (c) *M_1 and M_3 are each open with respect to I_n; and if M_1 and M_3 are each non-vacuous, then M_2 separates M_1 from M_3 in I_n.*

The following is an example of a continuous multi-valued transformation T of a 2-cell I_2 into itself, with the image of each point being a 1-sphere, such that M_2 is null and M_1 is a single point.

Using polar coordinates in the plane, let $I_2 = \{(r, \theta) \mid 0 \leq r \leq 1\}$. For $0 \leq s \leq 1$, define $T(s, 0)$ as the set of all (r, θ) such that (1) $r = 1$ or $r = 1 - s$ and $s \leq \theta \leq 2\pi - s$, or (2) $1 - s \leq r \leq 1$ and $\theta = s$ or $\theta = 2\pi - s$. For $0 \leq s \leq 1$ and $0 \leq \theta \leq 2\pi$, define $T(s, \theta)$ to be the set $T(s, 0)$ rotated through the angle θ.

In the proof of the above theorem two auxiliary functions S and W are defined on I_n into itself, where $S(P)$ is $T(P)$ together with its interior and $W(P)$ is the closure of $I_n - S(P)$. (Presumably $W(P)$ was meant to be $T(P)$ together with the intersection of the exterior of $T(P)$ with I_n.) The difficulty lies in the statement that S and W are continuous, which is false.

REFERENCE

1. O. H. HAMILTON, *A fixed point theorem for upper semi-continuous transformations on n-cells for which the images of points are non-acyclic continua,* this Journal, vol. 14(1947), pp. 689–693.

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Received August 29, 1956; presented to the American Mathematical Society November 30, 1956. This research was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command under contract No. AF 18(600)-1449.