CONVERSES OF SCHWARZ'S INEQUALITY

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1. Introduction. The well-known inequality of Schwarz states that

\[ \left( \int_0^1 u(x)v(x) \, dx \right)^2 \leq \left( \int_0^1 u^2 \, dx \right) \left( \int_0^1 v^2 \, dx \right), \]

for any two functions \( u \) and \( v \) belonging to \( L^2(0, 1) \). Without additional restrictions upon \( u \) and \( v \), there is no non-trivial inequality going in the other direction; i.e., one of the form

\[ \left( \int_0^1 u(x)v(x) \, dx \right)^2 \geq k \left( \int_0^1 u^2 \, dx \right) \left( \int_0^1 v^2 \, dx \right), \]

where \( k > 0 \).

If, however, \( u \) and \( v \) are restricted to lie within certain function classes, there do exist inequalities of the above form with \( k \) a positive constant dependent upon the classes chosen. The first discussion of problems of this variety occurs in the papers of Blaschke and Pick, [1], for the case where \( u(x) \) and \( v(x) \) are concave. (An earlier paper, Frank and Pick, Mathematische Annalen, vol. 76 (1915), p. 354, should also be consulted).

The purpose of this paper is to present a general method for attacking these problems which is equally applicable to other function classes and to multi-dimensional versions. Although it does not solve any particular problem completely, it reduces each problem to a particular investigation which in some cases can be carried through completely.

We shall begin with the one-dimensional case, demonstrating

**Theorem 1.** Let \( u(x) \) and \( v(x) \) be concave functions of \( x \) for \( 0 \leq x \leq 1 \), normalized by the conditions

\[ \int_0^1 u^2 \, dx = 1, \quad \int_0^1 v^2 \, dx = 1, \]

\[ u(0) = u(1) = 0, \quad v(0) = v(1) = 0. \]

Then

\[ \int_0^1 u(x)v(x) \, dx \geq \frac{1}{2}. \]

This result is certainly contained in the paper of Blaschke and Pick cited above, although apparently not explicitly stated in the above form. The minimum is attained for

\[ u(x) = x \sqrt{3}, \quad 0 \leq x < 1, \quad u(1) = 0, \]

\[ v(x) = (1 - x) \sqrt{3}, \quad 0 < x \leq 1, \quad v(0) = 0. \]

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