SOME CONNECTIONS BETWEEN TOPOLOGICAL AND ALGEBRAIC PROPERTIES IN RINGS OF OPERATORS

By Jacob Feldman

1. **Introduction.** Several theorems are herein proved which relate the maximal possible number of orthogonal projections in a ring of operators, the Murray-von Neumann "dimension" of the ring of operators, and the minimal possible cardinality of a weakly dense subset. When possible, this is done in the more general context of AW^* algebras. Theorem 1 states that a purely infinite AW^* algebra contains a purely infinite projection which is minimal, in the sense of the Murray-von Neumann partial ordering of equivalence classes of projections, among all the purely infinite projections whose central covers are equal to the identity; (this result is known for rings of operators; see, for example, [4]).

For the purpose of stating Theorems 2 and 3, some definitions. An AW^* algebra will be called *$\mathfrak{a}$-decomposable*, where $\mathfrak{a}$ is a cardinal number, if every set of nonzero orthogonal projections of the algebra has cardinality $\leq \mathfrak{a}$; it will be called *locally $\mathfrak{a}$-decomposable* if it is a C^* direct sum of $\mathfrak{a}$-decomposable summands; and it will be called *$\mathfrak{a}$-bounded* if every set of nonzero orthogonal equivalent projections has cardinality $\leq \mathfrak{a}$. $\aleph_0$-decomposability will be called, as is customary, *countable decomposability*. Then Theorem 2 asserts that every $\mathfrak{a}$-bounded AW^* albegra whose center is $\mathfrak{a}$-decomposable must itself be $\mathfrak{a}$-decomposable; and Theorem 3 states that a ring of operators with a weakly dense subset of cardinality $\leq \mathfrak{a}$ must be locally $\mathfrak{a}$-decomposable.

Familiarity with the contents of [2] is assumed.

2. **Some dimension theory.** Let $\mathbf{A}$ be a purely infinite AW^* algebra, $\mathbf{Z}$ its center, and f the canonical *-isomorphism from $\mathbf{Z}$ onto the algebra $\mathbf{C}(\Gamma)$ of all continuous complex-valued functions on the spectrum Γ of $\mathbf{Z}$. We construct a "dimension function" for purely infinite projections of $\mathbf{A}$: a map d assigning to each purely infinite projection P of $\mathbf{A}$ an order-continuous function $d(P)$ from Γ to cardinal numbers, such that $P \precsim Q$ if and only if $d(P)(\gamma) \leq d(Q)(\gamma)$ for all γ in Γ. This construction, based on a suggestion by I. Kaplansky, would seem on the face of it not to be the most natural one; for example, to the identity in the ring of all bounded operators on a separable Hilbert space it assigns the smallest uncountable cardinal, rather than the cardinal $\aleph_0$, thus differing from what one would ordinarily choose for a dimension function in a ring of operators; however, in our present state of knowledge about AW^* algebras, this seems to be the only possible choice for the purpose at hand.

For any infinite projection P in $\mathbf{A}$, define $h(P) = \wedge \{\mathfrak{a} \mid P$ cannot be split

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