BASIC PROPERTIES OF DISCRETE ANALYTIC FUNCTIONS

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1. Introduction. Of concern in this paper are complex-valued functions defined at the points of the complex plane whose coordinates are integers. These points form a lattice which breaks up the plane into unit squares. A function f is said to be discrete analytic at one of these squares if the difference quotient across one diagonal is equal to the difference quotient across the other diagonal,

$$\text{(c)} \qquad [f(z + 1 + i) - f(z)]/(1 + i) = [f(z + i) - f(z + 1)]/(i - 1).$$

If $f = u + iv$ where u and v are real, then it is seen that discrete analyticity implies that u and v satisfy a pair of difference equations which are analogous to the Cauchy-Riemann equations. If f is analytic in a region it results that u and v are discrete harmonic in that region. That is, they satisfy the Laplacian difference equation.

The above definition of analyticity was introduced by Jacqueline Ferrand (Lelong) [5]. She developed several interesting analogies with ordinary analytic functions. This paper extends her work in several directions. These new developments include analogies of the function z^{-1}, the Cauchy integral formula, Liouville's theorem, Harnack's inequality, polynomial expansions, and Hilbert transforms. Some of the developments here have no direct analogy in the classical continuous theory. These include the notion of duality, bipolynomials, and an operational calculus.

The theory and application of discrete harmonic functions have received considerable attention in the literature. Much of this work may be brought to bear on the present problem. In particular the paper of H. A. Heilbronn [6] concerning discrete harmonic polynomials and the paper of A. C. Allen and B. H. Murdoch [1] concerning the analog of Poisson's integral formula have been of value in the preparation of this paper.

Rufus Isaacs [7, 8] developed a theory of discrete analytic functions based on the following definition of analyticity:

$$\text{(a)} \qquad f(z + 1) - f(z) = [f(z + i) - f(z)]/i.$$

He preferred this definition to

$$\text{(b)} \qquad f(z + 1) - f(z - 1) = [f(z + i) - f(z - i)]/i,$$

which he also considered. It appears that definition (b) is essentially equivalent to the Ferrand definition (c). It is not apparent that (a) and (c) have any

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