# MULTIPLY MONOTONE FUNCTIONS AND THEIR LAPLACE TRANSFORMS

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**Introduction.** This paper considers a modification in two ways of the Bernstein-Widder representation of completely monotone functions on $(0, \infty)$:

$$f(t) = \int_0^\infty e^{-tu}\, d\beta(u) \tag{1}$$

where $\beta(u)$ is non-decreasing.

The analogue of (1) for $n$-times monotone functions, $n \geq 1$, is

$$f(t) = \int_0^\infty [(1 - ut)_+]^{n-1}\, d\beta(u), \tag{2}$$

with $\beta(u)$ non-decreasing. This formula was discovered by Schoenberg in 1940 but has remained unpublished. See, however, Schoenberg [11].

Part I consists of a discussion of (2) and its properties. In Part II the class $K_\alpha$ of $\alpha$-times monotone functions for arbitrary values of $\alpha$ greater than one is defined. The definition and theorems can be extended to values of $\alpha$ between zero and one, but, apparently, separate proofs are required, so this is not done here. In Part III the determining function of formula (1) is specialized to be absolutely continuous with its derivative in $K_\alpha$, and (1) is then considered both as a transform and as an analytic function of $t$.

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## PART I. THE $n$-TIMES MONOTONE FUNCTIONS.

### 1. Definitions and a lemma.

Definition 1. A function $f(t)$ is completely monotone for $t > 0$ if $(-1)^k f^{(k)}(t) \geq 0$ for $t > 0$ and for $k = 0, 1, 2, \cdots$.

Definition 2. A function $f(t)$ defined for $t > 0$ is $n$-times monotone where $n$ is an integer, $n \geq 2$, if $(-1)^k f^{(k)}(t)$ is non-negative, non-increasing and convex for $t > 0$, and for $k = 0, 1, 2, \cdots, n - 2$. When $n = 1$, $f(t)$ will simply be non-negative and non-increasing. Compare Royall [10].

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