A LIMIT THEOREM FOR THE MAXIMUM OF NORMALIZED SUMS OF INDEPENDENT RANDOM VARIABLES

By D. A. Darling and P. Erdös

1. **Introduction.** The main purpose of this paper is to prove the following theorem:

Theorem 1. *Let* $X_1, X_2, \cdots$ *be independent random variables with mean* 0, *variance* 1, *and a uniformly bounded third absolute moment. Put* $S_k = X_1 + X_2 + \cdots + X_k$ *and let*

$$U_n = \max_{1 \le k \le n} \frac{S_k}{k^{1/2}} \cdot \tag{1.1}$$

Then

$$\lim_{n\to\infty} \Pr\left\{U_n < (2 \log \log n)^{\frac{1}{2}} + \frac{\log \log \log n}{2(2 \log \log n)^{1/2}} + \frac{t}{(2 \log \log n)^{1/2}}\right\}$$
$$= \exp\left(-e^{-t}/2(\pi)^{\frac{1}{2}}\right), \qquad -\infty < t < \infty.$$

The corresponding limit theorem for $U_n' = \max_{1\le k\le n} S_k/n^{\frac{1}{2}}$ is well known [3], but the distribution of U_n is considerably more delicate, mainly because, speaking roughly, $S_k/k^{\frac{1}{2}}$ attains its maximum for a relatively small index, and the usual crude application of the central limit theorem will not work. Indeed, the above theorem is probably false if we drop the condition on the third absolute moment, even in the case of identically distributed X_i.

Theorem 1 solves, in an asymptotic form, the classical optional stopping problem (for example see Robbins [7]). Robbins gave a one-sided inequality for the distribution of $U_n'' = \max_{tn\le k\le n} S_k/k^{\frac{1}{2}}$, $0 < t < 1$, in the case of normally distributed X_i. In the case the X_i satisfy only the central limit theorem, Darling and Siegert [2] found the limiting distribution of U_n'' in terms of a Laplace transform, namely they found an explicit expression for

$$\int_0^\infty e^{-\lambda\sigma} \lim_{n\to\infty} \Pr\left\{\max_{e^{-2\sigma}n\le k\le n} \frac{S_k}{k^{1/2}} < \xi\right\} d\sigma.$$

The evaluation of the limiting distribution of U_n, given by (1.1), is however a qualitatively different matter.

This problem is also closely related to a problem posed by Levy [6] (footnote 19) on the law of the iterated logarithm for the Wiener process. Theorem 1 sheds some new light on the law of the iterated logarithm, and it may be true that the requirement on the third absolute moment could be replaced by the condition that the X_i are such that the law of the iterated logarithm holds.

Received June 24, 1955. This research was supported in part by the United States Air Force under Contract No. AF18(600)-685 monitored by the Office of Scientific Research.