MAXIMAL IDEALS
IN RINGS OF BOUNDED CONTINUOUS FUNCTIONS

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1. **Introduction.** Let X be a completely regular topological space, R the real numbers, and $C^*(X, R)$ the ring of all continuous bounded functions from X to R. Stone [6; Theorem 76] and others have shown that if M is any maximal ideal in C, then C^*/M is isomorphic to R. This result also holds if R is the complex numbers, quaternions, or a finite field. This theorem of Stone's is a basic result concerning the ring C^*, which has many important consequences, leading, among other things, to the Čech compactification theorem.

Kaplansky [2] has suggested the problem of attempting to extend Stone's theorem in the following way. Let A be an arbitrary topological division ring. Introduce the following definition (following Shafarevich [5] and Kaplansky [2]): a subset S of A is *bounded* if and only if for any neighborhood N of 0, there exist neighborhoods W_1 and W_2 of 0 such that $S \cdot W_1 \subset N$ and $W_2 \cdot S \subset N$. A function on a space X to A is bounded if and only if its range is a bounded set in A. Under this definition, let $C^*(X, A)$ be the ring of all bounded continuous functions from X to A. Can we prove that C^*/M is isomorphic to A for any maximal two-sided ideal M?

The purpose of this paper is to show that, with certain general hypotheses on our ring A and space X, the answer to this question is no. We deduce a necessary condition for Stone's theorem to hold, and show how to construct a maximal two-sided ideal M which, in the absence of this condition, has the property that C^*/M is not isomorphic to A. We conjecture that our condition is also sufficient, but we have not succeeded in proving this.

2. **Definitions and preliminary lemmas.** In this section we state our fundamental definitions, and extend the concept of boundedness which was mentioned above. We also obtain some simple lemmas which are needed for our purposes.

Definition 1. A is a *topological ring* if and only if A is a ring and a Hausdorff space, such that $a - b$ and ab are continuous functions of a and b, for $a, b \in A$.

In the following, A will always denote such a ring. The set of all neighborhoods of 0 will be denoted by Σ.

Definition 2. Let $S \subset A$. S is *bounded* if and only if $N \in \Sigma$ implies there exist $W_1, W_2 \in \Sigma$ such that $S \cdot W_1 \subset N$, $W_2 \cdot S \subset N$.

It is to be noted that if A is a metric ring, boundedness with respect to the metric implies boundedness as above defined. If A has a valuation, the two concepts are equivalent.

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