A NOTE ON GENERALIZED DEDEKIND SUMS

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1. **Introduction.** Rademacher [3] has recently proved a three-term relation for the Dedekind sum

$$s(h, k) = \sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right), \tag{1.1}$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}). \end{cases}$$

If a, b, c are positive integers such that $(a, b) = (b, c) = (c, a) = 1$, and $aa' \equiv 1 \pmod{bc}$, $bb' \equiv 1 \pmod{ca}$, $cc' \equiv 1 \pmod{ab}$, then

$$s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right). \tag{1.2}$$

The method of proof is that of [1].

In the present note we consider a sum that generalizes (1.1). In order to simplify the final formulas we take

$$f\left(\frac{r}{k}\right) = \frac{r}{k} - \left[\frac{r}{k}\right] - \frac{1}{2} + \frac{1}{2k} \tag{1.3}$$

and define

$$s_n(h_1, \cdots, h_n; k) = \sum_{r_1, \cdots, r_n \pmod k} f\left(\frac{r_1}{k}\right) \cdots f\left(\frac{r_n}{k}\right) f\left(\frac{r_1 h_1 + \cdots + r_n h_n}{k}\right), \tag{1.4}$$

for arbitrary $n \geq 1$; thus for $n = 1$, (1.4) is essentially the same as (1.1). Then we show that the sum s_n satisfies an $(n + 1)$-term relation and also an $(n + 2)$-term relation. The latter includes (1.2) when $n = 1$.

2. **A polynomial identity.** Let $k_1, \cdots, k_n$ denote n positive integers that are relatively prime in pairs. Put

$$g_i(x) = \frac{x^{k_i} - 1}{x - 1} \qquad (i = 1, 2, \cdots, n) \tag{2.1}$$

and

$$G(x) = \prod_{i=1}^{n} g_i(x), \qquad g_i(x) G_i(x) = G(x). \tag{2.2}$$

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