A CHARACTERIZATION OF THE MODULAR GROUP AND CERTAIN SIMILAR GROUPS

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The modular group, which is certainly the most studied group of conformal mappings of the upper half-plane onto itself, is the first of a sequence of discontinuous groups G_n, $n = 3, 4, 5, \cdots$, defined as follows: G_n is that group of conformal mappings of the upper half z-plane onto itself which is generated by the two transformations $z \to -z^{-1}$ and $z \to z + 2 \cos (\pi/n)$. Each of the groups G_n has a fundamental domain having finite hyperbolic area. (One may take that part of the upper half-plane exterior to the unit circle and between the parallel lines $\text{Re } z = -\cos \pi/n$ and $\text{Re } z = +\cos \pi/n$.)

Let the free product of a cyclic group of order two and one of order n be denoted by $Z_2 * Z_n$. Then it can be demonstrated by the methods of the author's paper [1], that G_n is algebraically isomorphic to $Z_2 * Z_n$, a matter already known for the modular group, G_3. The purpose of this paper is to prove the following theorem.

Theorem 1. *If a discontinuous group of conformal mappings of the upper half-plane onto itself is isomorphic to $Z_2 * Z_n$ and has a fundamental domain having finite hyperbolic area, then it is conjugate to G_n within the full group of conformal mappings of the upper half-plane onto itself.*

1. **Some notation.** If R is any Riemann surface, then a conformal mapping of R onto itself will be called an *automorphism.* If (σ_i) is a set of automorphisms of R, the Riemann surface whose points are the orbits of points under the group generated by the σ_i (provided the group is discontinuous) will be denoted by $R/(\sigma_i)$, and if (σ_i) contains but one element σ, simply by R/σ. The orbit of a point or set of points under a group of automorphisms will be called an orbit of the group.

The upper half-plane will always be denoted by M. A discontinuous group of automorphisms of M will be called *fine* if its fundamental domain has finite hyperbolic area, otherwise it will be called *gross.* This does not depend on the choice of fundamental domain, Siegel [5]. If H is an abstract group and G a discontinuous group of automorphisms of M isomorphic to it, then G will be called a *realization* of H and will be called fine or gross if G is fine or gross respectively. If G and G' are two realizations of H such that G is conjugate to G' within the full group of automorphisms of M, then these realizations will be considered not distinct.

In this terminology, Theorem 1 states that $Z_2 * Z_n$, $n = 3, 4, 5, \cdots$, has one and only one fine realization. It will also be shown that $Z_2 * Z_2$ has no

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