THE STRUCTURE OF NON-SEMISIMPLE ALGEBRAS

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Introduction. The main result of this paper is an extension of the Wedderburn-Malcev theorem for associative algebras to certain infinite dimensional algebras. The assumptions that the algebra $\mathfrak{A}$ be finite dimensional and that $\mathfrak{A}/\mathfrak{N}$ be separable are replaced by the requirements that the intersection of the powers of the Jacobson radical $\mathfrak{N}$ is the zero ideal, that the algebra is complete in the natural topology determined by the powers of $\mathfrak{N}$, and that the semi-simple residue class algebra $\mathfrak{A}/\mathfrak{N}$ is finite dimensional and separable over the base field. A number of examples can be found in the literature which show that some sort of restrictive hypotheses are required for the validity of the theorem, one of the most illuminating of which has been given by Zelinsky [13].

In §2 a class of rings is discussed, which in the commutative case reduce to the semi-local rings defined by Chevalley [3]. A sufficient condition for the completeness of a non-commutative extension of a complete semi-local ring is obtained. This result and the extension of the Wedderburn-Malcev theorem are applied in the last section to prove a theorem supplementary to an interesting result proved recently by Azumaya.

1. **An extension of the Wedderburn-Malcev theorem.** We begin with some preliminary results. The author is indebted to the referee for several important suggestions concerning these results, which lead to a great simplification of the author's original proof of part (i) of Theorem 1. Here, and elsewhere in the paper, we use the words "radical" and "semi-simple" in the sense of Jacobson [8].

Lemma 1. *Let $\mathfrak{A}$ be an algebra over a field K with radical $\mathfrak{N}$. Suppose that $\mathfrak{N}^2 = (0)$, and that $\mathfrak{A}/\mathfrak{N}$ is finite dimensional and separable over K. Then there exists a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B} + \mathfrak{N} = \mathfrak{A}$ and $\mathfrak{B} \cap \mathfrak{N} = (0)$.*

For a proof of this result we refer to [1; 47, 48]; it must be pointed out, however, that the results upon which the argument depends [1; Lemmas 3.4, 3.5, and 3.6] are valid in an algebra satisfying the hypothesis of the lemma, or more generally in any ring whose radical is a nil ideal [10; 32].

An alternative approach to Lemma 1 was suggested to the author by Professor N. Jacobson. In the terminology of Hochschild [7], Lemma 1 is precisely the statement that a finite dimensional separable algebra is segregated in every singular extension (Hochschild's proof is valid whether or not the kernel associated with the extension, which in this case is the radical $\mathfrak{N}$, is finite dimensional.) In fact, if $\mathfrak{B}$ is a separable algebra, then $H^1(\mathfrak{B}, \mathfrak{P})$ is zero for any two-sided $\mathfrak{B}$-module $\mathfrak{P}$, and it follows by the Reduction Theorem [7; Theorem 3.1] that

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