# AN ELEMENTARY PROOF OF A THEOREM OF JACOBSON

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In his paper "*Structure theory for algebraic algebras of bounded degree*" [3; Theorem 11] Jacobson proved the following very striking and beautiful theorem: *let $R$ be a ring such that for every $x \in R$ there exists an integer $n(x) > 1$, which depends on $x$, such that $x^{n(x)} = x$; then $R$ is commutative.* This result can be regarded, in a very natural way, as a generalization of the theorem of Wedderburn which states that a finite division ring is a commutative field.

In this paper we give a proof of Jacobson's theorem which rests strongly on the Wedderburn theorem. This is all done in a completely elementary way.

The first half of the paper is devoted to the establishment of Jacobson's theorem for the case that the ring is a division ring $K$. The proof here falls into two distinct parts:

(1) we show that if $xax^{-1} = a^r$ in $K$, then $a^r = a$ and $xa = ax$;

(2) using almost verbatim a simple and elegant argument due to Artin [1] we characterize the situation where two elements of $K$ satisfy the same minimal polynomial over the center. From this we are able to exhibit an $x$ and $a$ in $K$ satisfying $xax^{-1} = a^r \neq a$ in case $K$ should not be commutative.

The second half of the paper obtains an elementary reduction of the theorem for general rings to the theorem for the division rings. Using the Jacobson structure theory such a reduction can of course, be obtained quite handily. (A simple reduction using subdirect sums and subdirectly irreducible rings is given in a paper by Forsythe and McCoy [2].) However we avoid all such structure theory here. ($Z$ will always denote the center.)

1. In this section we assume that $K$ is a division ring in which $x^{n(x)} = x$ for all $x \in K$. We begin with

LEMMA 1.1. *$K$ is of characteristic $p \neq 0$.*

For suppose $x^n = x$, $(2x)^m = 2x$, with $n, m > 1$. Let $s = (n - 1)(m - 1) + 1$. Then $x^s = x$, $(2x)^s = 2x = 2^s x^s = 2^s x$. Hence $(2^s - 2)x = 0$, proving the lemma.

Let $P \subset Z$ be the prime field of $K$. Thus $P$ has $p$ elements. Suppose that for some $x, a \in K$, $xax^{-1} = a^r$. Let

$$T = \left\{ b \in K \mid b = \sum_{i=0}^{n(a)-2} \sum_{j=0}^{n(x)-2} p_{ij} a^i x^j, \quad p_{ij} \in P \right\}.$$

$T$ is clearly a finite set. Moreover $T$ is closed under addition. Since $xa = a^r x$, $T$ is also closed under multiplication. $T$ is thus a finite subring of $K$. Since the inverse of any element in $K$ is a power of the element itself, every element of $T$

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