IDEALS IN A CLASS OF COMMUTATIVE BANACH ALGEBRAS

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In the following, A shall always mean a commutative semi-simple Banach algebra, M the space of regular maximal ideals of A and $\bar{A}$ the Banach space conjugate to A. For $f \varepsilon A$, $\phi \varepsilon \bar{A}$, (f, ϕ) means the value at f of the functional ϕ. If A' is a subspace of A such that $(f, \phi) = 0$ for all $f \varepsilon A'$, we write $\phi(A') = 0$. If $f \varepsilon A$ and $m \varepsilon M$, $f(m)$ denotes the value at m of the complex-valued function on M associated to f by the Gelfand mapping. If I is an ideal of A, $h(I)$, the "hull" of I, means the set of $m \varepsilon M$ with $I \subseteq m$. $h(I)$ is evidently a closed set.

In this note we study certain ideals in a class of Banach algebras which includes the group-algebras of locally compact Abelian groups.

Definition. A has property (1) if given any m_0 in M and any neighborhood of m_0, there is some $f \varepsilon A$ with $f(m_0) = 1$ and $f(m) = 0$ outside the given neighborhood. An algebra satisfying (1) is called "regular".

Definition. A has property (2) if the set of $f \varepsilon A$ with $f(m)$ vanishing outside some compact set is dense in A.

Definition. $G(A)$ denotes the family of linear isometric operators U taking A onto A with $Ux \cdot y = Uy \cdot x$ for any $x, y \varepsilon A$.

Definition. A has property (3) if every hyperplane of A which is taken into itself by every U in $G(A)$ is an ideal in A.

It is well-known that a group algebra satisfies (1) and (2). Also in this case every translation operator, $f(x)$ into $f(x + t)$, is in $G(A)$ and every closed translation invariant subspace is an ideal. Hence a group algebra also satisfies (3). We shall prove:

Theorem 1. *Let A satisfy (1), (2) and (3). If I is a closed ideal in A included in precisely one maximal ideal m_0, then $I = m_0$.*

Theorem 2. *Let A satisfy (1), (2) and (3). Suppose in addition that for $f \varepsilon A$, $\epsilon > 0$ there is some $g \varepsilon A$ with $\| fg - f \| < \epsilon$. Let I be a closed ideal such that the boundary of $h(I)$ has no perfect nonempty subset. Then I is exactly the intersection of maximal ideals containing it.*

A group algebra satisfies the additional condition of Theorem 2 since a group algebra contains in fact a directed system e_λ with $e_\lambda f$ converging to f for every f in the algebra.

Theorem 2′. *Let A have a unit and a set of generators x which have inverses and are such that $\| xy \| = \| y \|$ for all $y \varepsilon A$. Then the conclusion of Theorem 2 remains true.*

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