CONVEX SETS IN LINEAR SPACES. III.

By V. L. Klee, Jr.

Introduction. (The first two papers of this series are [3] and [4]. We continue here the notation and terminology of [3].) Suppose L is a real linear system and let T be the collection of all sets $X \subset L$ such that for each finite-dimensional linear variety $V \subset L$, $X \cap V$ is open in the natural "Euclidean" topology of V. We show below that the topology T is locally convex if and only if $\dim L \leq \aleph_0$; and for $\dim L > \aleph_0$, L actually contains a closed convex proper subset which intersects every nonempty open convex set. This answers the question (Q_2) of [3]. (The *dimension* $\dim L$ of L is the cardinal number of a Hamel basis for L. When the contrary is not specified, topological adjectives will all refer to the topology T. For example, the open subsets of L are precisely those which belong to T.)

§2 contains two theorems on p^+-functionals, supplementing results of [3], and §3 contains some remarks on the strongest Hausdorff linear topology of a linear system. §4 discusses convex Borel sets and answers (Q_4) of [3] for the Euclidean plane.

1. **The linear space** $(L; T)$. Let H be a Hamel basis for L and let S be the linear system of all real-valued functions η on H such that $\eta(h) = 0$ for all but finitely many $h \in H$. For each $h \in H$, let γ_h be the characteristic function (on H) of $\{h\}$. Then the mapping $h \rightarrow \gamma_h$ generates an isomorphism of L onto S so that we may henceforth confine our attention to S.

A subset of a linear system will be called *linearly bounded* if its intersection with each line is bounded. (This notion originated with W. D. Berg and O. M. Nikodým.) The cardinal number of a set Z will be denoted by card Z.

For each nonempty finite $F \subset H$, let α_F be defined as follows: on $H \sim F$, $\alpha_F \equiv 0$; on F, $\alpha_F \equiv (\text{card } F)^{-2}$. For each set $U \subset H$, let $X_U = \{\alpha_F \mid \text{finite } F \subset U\}$.

(1.1) *Suppose U is an uncountable subset of H and $X = X_U$. Then* (i) *$X \cap V$ is finite for each finite-dimensional variety V;* (ii) *conv X is closed and linearly bounded;* (iii) *X intersects every open convex set which contains ϕ.*

Proof. For each $J \subset H$, let $S_J = \{\eta \mid \eta \in S \text{ and } \eta \equiv 0 \text{ on } H \sim J\}$. It is clear that for subsets A and B of H, $X_A \cap S_B = X_{A \cap B}$. But X_F is finite for each finite $F \subset H$ and each finite-dimensional variety V is contained in some X_F, so (i) must be satisfied. Since each α_F is somewhere positive and everywhere nonnegative, $S_J \cap \text{conv } Y = \text{conv } (S_J \cap Y)$ (for $J \subset H$ and $Y \subset S$).

Received May 21, 1952, in revised form, July 19, 1952. This paper was written while the author was a National Research Fellow at the Institute for Advanced Study.