NOTE ON IRREDUCIBILITY OF THE BERNOULLI AND EULER POLYNOMIALS

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1. **Introduction.** In the usual notation the Bernoulli and Euler polynomials may be defined by means of

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_m(x), \qquad \frac{2e^{xt}}{e^t + 1} = \sum_{m=0}^{\infty} \frac{t^m}{m!} E_m(x),$$

respectively. It is well known [2, Ch. 2] that for m odd ≥ 3, $B_m(x)$ has the three linear factors x, $x - \frac{1}{2}$, $x - 1$; for m even ≥ 2, $E_m(x)$ has the factors x, $x - 1$; for m odd, $E_m(x)$ has the factor $x - \frac{1}{2}$. Beyond this there seems to be nothing known about factorization of $B_m(x)$ and $E_m(x)$ relative to the rational field.

In the present note we collect a few fragmentary results in this direction. Let p denote a prime ≥ 3. We show that $B_{m(p-1)}(x)$ is irreducible for $1 \leq m \leq p$; also $B_m(x)$ is irreducible for $m = 2^r$ and $m = k(p-1)p^r$, $1 \leq k < p$. In the case of an odd index we show that $B_{2m+1}(x)/x(x - \frac{1}{2})(x - 1)$, where $2m + 1 = k(p - 1) + 1$, $k \leq p$, if not itself irreducible has an irreducible factor of degree $\geq 2m + 1 - p$.

For the Euler polynomials the situation is even more obscure. If $p \equiv 3 \pmod 4$, then $E_p(x)/(x - \frac{1}{2})$ is irreducible; however, for $p \equiv 1 \pmod 4$, we can no longer make the same assertion. Indeed

$$E_5(x) = (x - \tfrac{1}{2})(x^4 - 2x^3 - x^2 + 2x + 1) = (x - \tfrac{1}{2})(x^2 - x - 1)^2,$$

so that repeated factors cannot be ruled out. We remark that

$$E_4(x) = x(x - 1)(x^2 - x - 1);$$

thus $E_4(x)$ and $E_5(x)$ have a common non-trivial factor. We shall show that $E_{2p}(x)/x(x - 1)$ has an irreducible factor of degree $\geq p - 1$.

The Bernoulli number of order k is defined by means of

$$\left(\frac{t}{e^t - 1}\right)^k = \sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!}. \tag{1.1}$$

It follows that $B_m^{(x)}$ is a polynomial in x of degree m. We shall show that $B_{p-1}^{(x)}/x$ is irreducible.

From the above it appears that irreducibility questions concerning the Bernoulli and Euler polynomials are somewhat similar to those involving the Legendre polynomials (see, for example, a recent paper by J. H. Wahab [4]).

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