A TOPOLOGICAL CHARACTERIZATION OF A CLASS OF AFFINE TRANSFORMATIONS

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1. **Introduction.** Let the symbols $\{x_\sigma\}_m$ and $\{y_\sigma\}_m$ denote real-valued sequences of period m, $m \geq 2$; *i.e.*,

$$x_{\sigma+m} = x_\sigma, \qquad y_{\sigma+m} = y_\sigma \qquad (\sigma = 0, \pm 1, \pm 2, \cdots).$$

Any pair of such sequences defines in the xy-plane the sequence of points (x_σ, y_σ) $(\sigma = 1, 2, \cdots, m)$. A closed oriented polygon can be defined by these points as follows:

(a) construct the horizontal line segments with the end-points

$$(x_\sigma, y_\sigma) \text{ and } (x_{\sigma+1}, y_\sigma) \qquad (\sigma = 1, 2, \cdots, m),$$

(b) construct the vertical line segments with the end-points

$$(x_{\sigma+1}, y_\sigma) \text{ and } (x_{\sigma+1}, y_{\sigma+1}) \qquad (\sigma = 1, 2, \cdots, m),$$

(c) give the polygon the orientation induced by the indices.

A sequence of m points will be said to *determine* the closed oriented polygon described above. A closed oriented curve is said to be of *non-negative circulation* if the order, with respect to the curve, of every point not on the curve is non-negative [3].

Consider the transformation

$$y_\sigma = -\sum_{\rho=1}^{m} a_\rho x_{\sigma-\rho+1}, \tag{1}$$

where the a_ρ are terms of an arbitrary real-valued sequence of period m, $\{a_\rho\}_m$. The transformation (1) maps a periodic sequence $\{x_\sigma\}_m$ into the periodic sequence $\{y_\sigma\}_m$. These two sequences, as previously described, define a sequence of points which in turn determine a closed oriented polygon. For convenience, it will be said that, given a sequence $\{x_\sigma\}_m$, the transformation (1) *generates* a polygon. The principal theorem can now be stated.

Theorem. *A necessary and sufficient condition that the transformation* (1) *generate only polygons of non-negative circulation for all sequences* $\{x_\sigma\}_m$ *is that there exist non-negative* C_ρ, β_ρ *and* γ_ρ *such that*

$$a_{k+1} - a_k = \sum_{\rho=1}^{q} C_\rho(\beta_\rho)^{m-k-1}(\gamma_\rho)^{k-1} \qquad (k = 1, 2, \cdots, m-1),\ q = \left[\frac{m}{2}\right].$$

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