INVARIANT MEASURE AND THE ERGODIC THEOREMS

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1. Introduction. Let there be given a measure space of finite measure and a measurable $1 - 1$ point transformation of the space onto itself. G. D. Birkhoff [2] and J. von Neumann [7] have proved the individual ergodic theorem and mean ergodic theorem (see Theorems 1 and 2 below) under the assumption that the measure is invariant with respect to the transformation. It is quite easy to see that the above mentioned theorems hold with certain modifications, not only when the given measure is invariant, but also when there exists an invariant measure defined for all the measurable sets of the space and having no more null sets than the given measure. If such an invariant measure exists we say that the given measure is potentially invariant. The question arises whether conversely the given measure is potentially invariant when either of the ergodic theorems hold. In this paper we answer this question in the affirmative by constructing in either case an invariant measure with the required properties. (See Theorems 5 and 6 below.) Quite generally, without making any assumption about the underlying measure, one can define a finitely additive, non-negative invariant set function on the measurable sets of the space by using the obvious averaging process and the Banach-Mazur limit. This set function is in general not unique, nor is it countably additive. In case the underlying measure is potentially invariant, and only in this case, this set function is countably additive, unique and has no more null sets than the given measure. This is the case if either of the ergodic theorems holds. This last result follows from a more general sufficient condition for the measure to be potentially invariant, a condition which deals with convergence in measure of averages of functions. (See Theorem 4 below.) We also give another sufficient condition dealing with averages of measures of sets. (See Theorem 7 below.)

Some of the methods used in this paper are similar to those used by K. Yosida [10]. The reader will also find points of similarity between our methods and results and those published in a recent paper by N. Dunford and D. S. Miller [3]. While Dunford and Miller deal with more general transformations, in the special case of a $1 - 1$ transformation and its iterates some of our results are generalizations of theirs.

2. Notations and basic concepts. Let (S, F, m) be a measure space where S is an abstract set, F is a Borel family of subsets of S and $m(A)$ is a countably additive non-negative set function defined for all sets $A \in F$. We shall call the elements x of S points the elements of F measurable sets and the number $m(A)$

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