A GENERALIZATION OF SCHWARZ' LEMMA

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The object of this paper is to give a generalization of Schwarz' Lemma for the case of non-uniform functions defined in the unit circle, or rather on a many-sheeted Riemann domain covering the unit circle. How this generalization is to be applied to various problems in the theory of functions will be shown in a number of examples.

The theorem to be proved may be stated as follows.

Theorem 1. *Let $w = f(z)$ be a non-uniform function, regular for $|z| < 1$ apart from a finite number of algebraic branch-points, and let $f'(z)$ be finite everywhere in $|z| < 1$; let further, for all determinations of $f(z)$, $|f(z)| \leq 1$ for $|z| <$ 1. Then we have*

$$|f'(0)| \leq 1$$

for all the different values $f'(0)$ may assume. The case $|f'(0)| = 1$ can only happen for $f(z) \equiv Kz$, $|K| = 1$.

Remark. The assumption that $f'(z)$ be finite everywhere in $|z| < 1$ is unavoidable; if this hypothesis is dropped, the theorem is not necessarily true, as is shown by the function $g(z)$ defined by

$$z = g(z)\left[\frac{\alpha - g(z)}{1 - \alpha^* g(z)}\right] \qquad (|\alpha| < 1),$$

where α^* denotes the complex conjugate of α, which satisfies all the other conditions and for which $|g'(0)| = |\alpha|^{-1} > 1$.

Proof. The demonstration of Theorem 1 will be based on a method of proof devised by Ahlfors in connection with an extension of Schwarz' Lemma for uniform functions [1].

The function $w = f(z)$ is uniform on a Riemann domain R consisting of a finite number of sheets; all boundary continua of R coincide with $|z| = 1$. If u and v denote the functions

$$(1) \qquad \begin{aligned} u &= \log \frac{|f'(z)|}{1 - |f(z)|^2}, \\ v &= \log \frac{r}{r^2 - |z|^2} \qquad (r < 1), \end{aligned}$$

then everywhere on R, with the exception of the points at which $f'(z) = 0$, we have

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