EXPANSION THEOREMS OF PALEY-WIENER TYPE

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1. Let $\{f_n\}$ and $\{g_n\}$ be two sequences in a Hilbert space H and suppose that for some number θ ($0 \leq \theta < 1$), and for all corresponding finite linear combinations $\varphi = \sum a_n f_n$, $\psi = \sum a_n g_n$, we have

$$\| \varphi - \psi \| \leq \theta \| \varphi \|. \tag{1}$$

Then,

1. *if* $\{f_n\}$ *is a complete set, i.e. spanning the whole space* H, *so is* $\{g_n\}$ (completeness theorem of Boas [1]; he proves the theorem for arbitrary Banach spaces);

2. *if* $\{f_n\}$ *is a complete orthonormal set, then* $\{g_n\}$ *admits a bi-orthonormal sequence* $\{h_n\}$ *and every element* u *of* H *has the expansions*

$$u = \sum (u, h_n) g_n, \qquad u = \sum (u, g_n) h_n,$$

the coefficients satisfying the inequalities

$$\begin{aligned} (1 + \theta)^{-1} \| u \| &\leq \{\sum | (u, h_n) |^2\}^{\frac{1}{2}} \leq (1 - \theta)^{-1} \| u \|, \\ (1 - \theta) \| u \| &\leq \{\sum | (u, g_n) |^2\}^{\frac{1}{2}} \leq (1 + \theta) \| u \| \end{aligned} \tag{2}$$

(*expansion theorem of Paley-Wiener* [2]).

Pollard has shown [3] that *the completeness theorem holds true even if* (1) *is replaced by the weaker condition*

$$\| \varphi - \psi \|^2 \leq \lambda_1 \| \varphi \|^2 + \lambda_2 \| \psi \|^2 \tag{3}$$

for some fixed λ_1, λ_2 ($0 \leq \lambda_1 < 1$, $0 \leq \lambda_2 < 1$).

As to the expansion theorem, Pollard remarks that "while our restrictions are less severe due to the presence of the λ_2 term, our conclusion is correspondingly weaker. For Paley and Wiener obtain also an expansion theorem, and this we lack."

Now, we shall show that the expansion theorem actually holds also under condition (3). Moreover, we shall show that *both theorems hold under the still more general condition*

$$\| \varphi - \psi \|^2 \leq \lambda \| \varphi \|^2 + 2\mu \| \varphi \| \| \psi \| + \nu \| \psi \|^2 \tag{4}$$

for some fixed λ, μ, ν ($0 \leq \lambda < 1$, $0 \leq \nu < 1$, $0 \leq \mu$, $\mu^2 \leq (1 - \lambda)(1 - \nu)$); *we have only to replace* (2) *by*

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