CAUCHY PRODUCTS OF DIVISOR FUNCTIONS IN $GF[p^n, x]$

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1. **Introduction.** Let $GF[p^n, x]$ denote the ring of polynomials in an indeterminate x with coefficients in the Galois field $GF(p^n)$. In this paper capital italics will denote polynomials in $GF[p^n, x]$ unless otherwise stated. By sgn A will be meant the coefficient of the highest power of x in A; if sgn $A = 1$, A is called *primary.*

A single-valued function $\phi(A)$ defined for all $A \in GF[p^n, x]$ will be called *arithmetic*; the values $\phi(A)$ are assumed to be complex numbers. Let ϕ and ψ be given arithmetic functions and F a given polynomial of degree f. Then we consider three types of composition, which will be referred to as Cauchy products C_1, C_2, C_3:

$$C_i: \quad \phi \cdot \psi \equiv \sum_i \phi(A)\psi(B) = \zeta(F) \qquad (i = 1, 2, 3). \tag{1.1}$$

In each case the summation is over polynomials A, B such that $A + B = F$, with the following restrictions:

Let r denote a fixed non-negative integer and α and β fixed non-zero elements of $GF(p^n)$, where $\alpha + \beta = \text{sgn } F$ if $f = r$ and $\alpha + \beta = 0$, sgn F arbitrary if $f < r$. Then under C_1, A and B range over polynomials of degree r with sgn $A = \alpha$, sgn $B = \beta$ and $A + B = F$. Under C_2, F is assumed $\neq 0$, of degree r, and the summation in (1.1) is over A of degree r and B of degree less than r such that $A + B = F$. Under C_3, F is assumed to be of degree less than r, and A and B range over polynomials of degree less than r such that $A + B = F$. By $\sum_i$ we shall mean a summation corresponding to C_i $(i = 1, 2, 3)$; a symbol such as $\sum_{2,3}$ will be used to indicate summations with respect to either C_2 or C_3.

The Cauchy products just defined are evidently analogous to the ordinary Cauchy product (see, for example, E. T. Bell [1]). However, as we shall see, there are important differences; in particular, in the polynomial case zero divisors occur—that is, $\zeta(F)$ in (1.1) may be identically zero, even though neither ϕ nor ψ is zero. For other properties see the end of §§2, 5.

In this paper we consider only a special class of arithmetic functions which we shall call divisor functions and which we shall now define. We first introduce certain notation to be used throughout the paper. If M denotes a polynomial in $GF[p^n, x]$, we define as in [2]:

$$\delta_z(M) = \begin{cases} \displaystyle\sum_{Z \mid M}^{\deg Z = z} 1 & (z \geq 0), \\ 0 & (z < 0), \end{cases} \tag{1.2}$$

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