AFFINE INVARIANTS OF CERTAIN PAIRS OF CURVES AND SURFACES

By L. A. Santaló

1. **Introduction.** For two curves in a plane or two surfaces in ordinary space various projective invariants have been given by Mehmke, Bouton, Segre, Buzano, Bompiani, Hsiung and others (see Bibliography).

Obviously each projective invariant is also an affine invariant, that is, an invariant with respect to the group of affine transformations. However in certain cases there are affine invariants which are not projective invariants. The purpose of the present paper is to study these cases giving affine invariants, as well as their affine and metrical characterization, for the following cases:

(a) two curves in a plane having a common tangent at two ordinary points (§§2, 3);

(b) two curves in a plane intersecting at an ordinary point (§§4, 5);

(c) two surfaces in ordinary space having a common tangent plane at two ordinary points (§§6, 7);

(d) two surfaces in ordinary space having a common tangent line but distinct tangent plane at two ordinary points (§§8, 9).

For the cases (a), (b) of plane curves we shall consider the neighborhoods of the second and the third order of the curves at the considered points. For the cases (c), (d) of two surfaces in ordinary space we shall consider only the neighborhoods of the second order of the surfaces at the considered points.

2. **Affine invariants of two plane curves having a common tangent at two ordinary points.** Suppose that O and O_1 are two ordinary points of two plane curves C and C_1 respectively, so that OO_1 is the common tangent. Let h be the distance OO_1. If we choose a cartesian coordinate system in such a way that the point O be the origin and the line OO_1 be the x-axis, the power series expansions of the two curves in the neighborhood of the points O and O_1 may be written in the form

$$C: \qquad y = ax^2 + bx^3 + \cdots, \tag{2.1}$$

$$C_1: \qquad y = a_1(x-h)^2 + b_1(x-h)^3 + \cdots, \tag{2.2}$$

where we suppose $a, a_1 \neq 0$.

In order to find the affine invariants of the elements of the second and the third order of the curves C, C_1 in the neighborhood of O, O_1 we have to consider the most general affine transformation which leaves the point O and the x-axis invariant:

$$x = \alpha X + \beta Y, \qquad y = \mu Y, \tag{2.3}$$

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