ABSOLUTE AND UNCONDITIONAL CONVERGENCE IN BANACH SPACES

BY M. E. MUNROE

Let us denote by I the class of real-valued functions defined on the closed interval [0, 1] which are indefinite Lebesgue integrals, by AC the class of real-valued functions which are absolutely continuous in [0, 1], by BV the class of real-valued functions of bounded variation in [0, 1], and by D the class of real-valued functions which are differentiable almost everywhere in [0, 1]. Then the following theorem is well known and contains what is usually referred to as the Fundamental Theorem of Integral Calculus.

THEOREM 0.1.
(a) $I = AC \subset BV \subset D$.
(b) *If an integral is almost everywhere differentiable, its derivative is almost everywhere equal to the integrand.*

If, instead of real-valued functions, we consider functions whose ranges lie in a fixed but arbitrary Banach space, part (b) of Theorem 0.1 holds for several well-known definitions of integration (see, for example, [9; 300–301]), but part (a) breaks down. Relations of the type stated in part (a) may hold, however, if the class of range spaces under consideration is suitably restricted. For example, Clarkson [5] has proved that a sufficient (but not necessary) condition that $BV \subset D$ is that the range space be uniformly convex. We shall investigate in this paper the nature of range spaces for which other such relations hold.

In particular, we shall consider the relations $AC \subset BV$ and $I = AC$, where AC and BV are defined in terms of the strong topology and I is the class of integrals defined by Bochner and Dunford (see [4], [6]). These questions lead to a consideration of the relation between absolute and unconditional convergence of series which in turn involves the discussion of a generalized notion of orthogonality. Sections 2 and 3 are concerned with convergence and orthogonality respectively.

The author stated in an abstract (Bulletin of the American Mathematical Society; abstract 48–7–241) that finite dimensionality of a Banach space was necessary and sufficient for the equivalence of unconditional and absolute convergence of series in the space. An error in his proof was discovered, and the paper was not published in full. The truth or falsity of this theorem remains an open question. Theorem 2.2 of the present paper throws some light on the matter, but fails to clear it up completely.

Received May 7, 1942; in revised form October 10, 1945 and March 31, 1946. Presented to the American Mathematical Society, September 8, 1942, under the title *On the finite dimensionality of certain Banach spaces*.