INVERSION FORMULAE FOR THE FACTORIAL TRANSFORM

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In a previous article [1] the author gave a discussion of some properties of the factorial transform, which was defined to be an integral of the form

$$\text{(A)} \qquad \int_0^\infty B(s, t+1)\, d\alpha(t).$$

It was shown that an integral of the form (A) could be written in the form of a Laplace transform

$$\text{(B)} \qquad \int_0^\infty e^{-sy}\, \phi(y)\, dy \qquad (\sigma = R(s) > \sigma_c),$$

where $\phi(y) = -\log(1 - e^{-y}) \int_0^\infty (1 - e^{-y})^t \alpha(t)\, dt$, and σ_c is the abscissa of convergence of the integral (A).

1. In this paper a pair of real inversion formulae for integrals of the form (A) will be derived. Both of these make use of the Phragmén [2] inversion formula for the Laplace transform. Phragmén's article assumes the existence of a finite abscissa of absolute convergence for the Laplace transform, but we shall show that the result is valid if only the convergence abscissa is finite. Throughout the paper it will be supposed that the functions $\alpha(t)$ which appear are normalized functions of bounded variation in every finite interval $(0, R)$.

Theorem. (Phragmén). *Let*

$$(1.1) \qquad f(s) = \int_0^\infty e^{-st}\, d\alpha(t),$$

the integral converging for $R(s) > \sigma_c$. *Then*

$$(1.2) \qquad \lim_{s \to +\infty} \sum_{n=1}^\infty \frac{(-)^{n+1}}{n!} f(ns)\, e^{nst}$$

equals $\alpha(t)$, *where* $\alpha(t)$ *is continuous, or equals* $(1 - 1/e)\alpha(t+) + (1/e)\alpha(t-)$, *where* $\alpha(t)$ *is discontinuous, with limits* $\alpha(t+)$ *and* $\alpha(t-)$ *on the right and left respectively.*

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