OPEN AND CLOSED TRANSFORMATIONS

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A single-valued transformation $f(A) = B$ will be said to be *closed* on A if every closed set in A maps into a closed set in B under f. If every open set in A maps into an open set in B, then f is said to be an *open* transformation on A. Thus a continuous open transformation is interior, and by analogy, a continuous closed transformation is *exterior*. A transformation $f(A) = B$, not necessarily continuous, is said to be *weakly non-alternating* provided that for no two points x, y of B does there exist a separation $A - f^{-1}(x) = A_1 + A_2$ such that $y \in f(A_1) \cdot f(A_2)$.

This paper is concerned with properties of the transformations defined above. The first section is devoted to closed and exterior transformations. A theorem, similar to the Eilenberg theorem for interior transformations, is given for open transformations in the second section. In the third, the action of open, weakly non-alternating transformations on arcs and simple closed curves is discussed. The relation of each of these transformations to continuity is given in the respective section. All spaces considered are separable metric.

1. Characteristic properties of open, closed, and exterior transformations.

(1.1) *The transformation $f(A) = B$ is exterior on A if and only if $f(R^*) = f^*(R)$, where R is any subset of A.*

(R^* and $f^*(R)$ denote the closure of R and $f(R)$, respectively.)

Proof. If f is exterior on A, $f(R^*)$ is closed in B, and $f^*(R) \subset f^*(R^*) = f(R^*)$. By continuity, $f(R^*) \subset f^*(R)$. Hence $f(R^*) = f^*(R)$.

Suppose $f(R^*) = f^*(R)$. Then f is continuous, and if X is any set closed in A, $f(X) = f(X^*) = f^*(X)$ is closed in B. Hence f is exterior on A.

(1.2) *A transformation $f(A) = B$ is open (closed) on A if and only if f is open (closed) on every inverse set Q of A.* (See [3; 147].)

Proof. Let f be closed on A. By definition $Q = f^{-1}f(Q)$. Let X_q be any subset of Q which is closed in Q. Then $X_q = X_q^* \cdot Q$. By hypothesis $f(X_q^*)$ is closed in B, and

$$f(X_q) = f(X_q^* \cdot Q) = f(X_q^*) \cdot f(Q) = f^*(X_q^*) \cdot f(Q) \supset f^*(X_q) \cdot f(Q) \supset f(X_q),$$

since Q is an inverse set in A (see [3; 146]). Therefore $f(X_q) = f^*(X_q) \cdot f(Q)$. Hence $f(X_q)$ is closed in $f(Q)$, and the statement follows.

The proof for the case where $f(A) = B$ is open is equally simple.

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