THE FRÉCHET DIFFERENTIALS OF REGULAR POWER SERIES IN NORMED LINEAR SPACES

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The existence and term by term Fréchet differentiability of all orders of a regular power series in *complex* Banach spaces (complete normed linear spaces with complex numbers as multipliers) was proved by Robert S. Martin [2] throughout the "sphere of analyticity" of the power series. The method of proof is inapplicable to power series in Banach spaces. A special method of proof suitable for a special power series in a special Banach space (a real normed linear ring) was given by Michal and Martin [5].

In this note we give a simple proof by entirely different methods of the general result for power series in Banach spaces. The need for such a theorem in many branches of analysis and differential geometry has been outstanding for many years. The author has already written two papers [3], [4] dealing with the applications of this theorem to several important problems in analysis.

Let $p_n(x)$ be a homogeneous polynomial of degree n on a Banach space E_1 to a Banach space E_2. Hence,

$$p_n(x + \lambda y) = p_n(x) + \lambda p_{n-1,1}(x, y) + \lambda^2 p_{n-2,2}(x, y) + \cdots + \lambda^{n-1} p_{1,n-1}(x, y) + \lambda^n p_n(y), \tag{1}$$

where $p_{n-r,r}(x, y)$ is homogeneous of degree $n - r$ in x and homogeneous of degree r in y. Clearly $p_{n-1,1}(x, y)$ is the first Fréchet differential $p_n(x; y)$ of $p_n(x)$ at $x = x$ with increment y.

Denote by $m(p_n)$ the modulus of $p_n(x)$ so that

$$\| p_n(x) \| \leq m(p_n) \| x \|^n. \tag{2}$$

Let $L(z)$ be any linear functional on E_2 to the real numbers with modulus unity. Define

$$f(\lambda) = L(p_n(x + \lambda y)). \tag{3}$$

If a prime denotes numerical differentiation, then from (1) and the composition theorem for Fréchet differentials we obtain

$$f'(0) = L(p_{n-1,1}(x, y)). \tag{4}$$

The following chain of inequalities holds for $| \lambda | \leq 1$:

$$| f(\lambda) | \leq \| p_n(x + \lambda y) \| \leq m(p_n) \| x + \lambda y \|^n \leq m(p_n)(\| x \| + \| y \|)^n. \tag{5}$$

To proceed further we need to prove a lemma in classical analysis.

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