FUNCTIONS OF EXPONENTIAL TYPE, V

BY R. P. BOAS, JR.

1. Although the connection between the distribution of the zeros and the growth of an entire function is, generally speaking, less precise for a function of integral order than for one of nonintegral order, very precise results are available for even functions of order one. This is because an even function of order one can be written as $g(z^2)$, where $g(z)$ is of order $\frac{1}{2}$. In this note I shall show that results for even functions can be extended to functions which are "almost even" (in a sense to be made precise below).

Consider a sequence of complex numbers $\{\lambda_n\}_{n=-\infty}^{\infty}$ satisfying

$$|\lambda_{n+1} - \lambda_n| > \delta > 0, \tag{1.1}$$

and

$$|\lambda_{-n} + \lambda_n| = O(n^\beta) \qquad (\beta < 1;\, n \to +\infty). \tag{1.2}$$

We denote by $n(r; \theta_1, \theta_2)$ the number of λ_n with $|\lambda_n| \leq r$ and $\theta_1 \leq \operatorname{arc} \lambda_n < \theta_2$, and we suppose that there is an increasing function $n(\theta)$ such that $n(\theta) = \frac{1}{2}[n(\theta+) + n(\theta-)]$, $n(\theta + 2\pi) = n(\theta) + n(2\pi)$, and

$$\lim_{r\to\infty} r^{-1} n(r; \theta_1, \theta_2) = n(\theta_2) - n(\theta_1). \tag{1.3}$$

This condition implies that the λ_n are quite uniformly distributed in each angle. We consider first the case where the λ_n are also distributed with the same density in every angle.

THEOREM 1. *Let $\{\lambda_n\}$ satisfy* (1.1) *and* (1.2) *and let $n(\theta) = L\theta$. Then*

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)\left(1 - \frac{z}{\lambda_{-n}}\right) \tag{1.4}$$

is an entire function of order 1 *and type* $2\pi L$.

The content of Theorem 1 can be roughly expressed by saying that if the zeros of $f(z)$ are uniformly distributed over the plane, then the type of $f(z)$ is as small as the density of its zeros will permit. In fact, the zeros of $f(z)$ have density $2\pi L$, and the type is not less than the density of the zeros (see, for example, [4; vol. 2, p. 10, problem 63]).

A theorem of Estermann [1; 259] states that if $f(z)$ is of order 1 and type τ, $\tau < 1$, and if $f(z_n) = 0$ with $|z_n| = n$ $(n = 1, 2, \cdots)$, then $f(z) \equiv 0$. Theorem 1 shows that $\tau < 1$ cannot be replaced by $\tau = 1$; for example, the sequence (suggested by N. Levinson) $(k^2 + l)e^{\pi i l/k}$ $(l = 0, 1, \cdots, 2k;\, k = 1, 2, \cdots)$

Received June 5, 1945.