A THEOREM ON FRACTIONAL DERIVATIVES

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1. A function $f(x)$ defined in the neighborhood of a point x_0 is said to possess at that point a *generalized derivative* of order k, if there exist constants α_0, α_1, α_2, $\cdots$, α_k such that the term $\epsilon(t)$ defined for small $|t|$ by the equation

$$f(x_0 + t) = \alpha_0 + \frac{\alpha_1}{1!} t + \cdots + \frac{\alpha_{k-1}}{(k-1)!} t^{k-1} + \frac{\alpha_k + \epsilon(t)}{k!} t^k \tag{1.1}$$

tends to 0 with t. The number α_k is called the k-th *generalized derivative* of f at the point x_0 and will be denoted by $f_{(k)}(x_0)$. The existence of $f_{(k)}(x_0)$ implies that of $f_{(k-1)}(x_0)$ (and so also of $f_{(k-2)}(x_0)$, $\cdots$, $f_{(1)}(x_0)$) but at individual points x_0 there exists no relation between $f_{(k)}(x_0)$ and $f_{(k-1)}(x_0)$. If $f^{(k)}(x_0)$ exists, so does $f_{(k)}(x_0)$ (and both numbers are equal). For $k = 1$, but not for $k > 1$, the converse is also true.

Suppose now, to fix the ideas, that $f(x)$ is defined for $x \geq 0$, and that it is integrable L over any finite interval $(0, b)$. If α is any positive number, the α-th integral $f_\alpha(x)$ of f will be defined by the formula

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t)\, dt. \tag{1.2}$$

Instead of $f_\alpha(x)$, we shall also write $I_\alpha[f]$.

If $0 < \alpha < 1$, the function $f_\alpha(x)$ exists for almost every x and is integrable over any finite interval. If $\alpha \geq 1$, $f_\alpha(x)$ exists everywhere and is continuous (indeed, absolutely continuous). If α is a positive integer, $f_\alpha(x)$ is the α-th primitive of f fixed by the condition that at the point $x = 0$ it vanishes along with all its derivatives of order $<\alpha$.

Suppose now that α is both positive and fractional. Thus $p - 1 < \alpha < p$, where p is a positive integer. Let $g(x) = I_{p-\alpha}[f]$. Suppose that $g(x)$ has at a point x_0 a p-th generalized derivative $g_{(p)}(x_0)$. We shall then say that $f(x)$ has at the point x_0 an α-th generalized derivative $f_{(\alpha)}(x_0)$, the latter being defined by the formula

$$f_{(\alpha)}(x_0) = g_{(p)}(x_0). \tag{1.3}$$

We might also define $f_{(\alpha)}(x_0)$ as $g^{(p)}(x_0)$, provided the latter number exists, and this would be the usual definition of fractional derivative. For our purposes, however, the definition (1.3) is more appropriate. If $0 < \alpha < 1$, both definitions coincide.

It is easy to see that if $0 < \alpha' < \alpha$, and if $f_{(\alpha)}(x)$ exists, so does $f_{(\alpha')}(x_0)$.

A very well-known result of Weyl [3] asserts that if $f(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$,

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