ORTHOGONALITY IN NORMED LINEAR SPACES

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Numerous equivalent definitions of orthogonality of elements of abstract Euclidean spaces can be given. The most natural is: "$x \perp y$ if and only if the inner product (x, y) is zero". This concept of orthogonality has certain desired properties, the most important being that every two dimensional subspace contains nonzero orthogonal elements. Two definitions of orthogonality in normed linear spaces will be given which preserve this property. Relations between orthogonal elements will be studied and it will be shown that for either definition the assumption of properties of homogeneity or additivity of the orthogonality implies the space is an abstract Euclidean space.

1. Normed linear spaces and abstract Euclidean spaces.

A *linear* (*vector*) *space* [1; 26] is an abstract Abelian group for which multiplication by real numbers is defined and satisfies the postulates:

1. $a(x + y) = ax + ay$ for all elements x and y and numbers a.
2. $(a + b)x = ax + bx$.
3. $a(bx) = (ab)x$.
4. $1 \cdot x = x$.

It follows that $0 \cdot x = 0$ and that: If $ax = 0$ and $a \neq 0$, then $x = 0$; if $ax = ay$ and $a \neq 0$, then $x = y$; if $ax = bx$ and $x \neq 0$, then $a = b$.

A linear space is a *normed linear space* [1; 53] if to each element x of the space there is associated a real number $\| x \|$ satisfying the postulates:

5. $\| x \| > 0$ if $x \neq 0$.
6. $\| ax \| = | a | \, \| x \|$ for all numbers a.
7. $\| x + y \| \leq \| x \| + \| y \|$.

A normed linear space is a *metric space* [1; 8] if the distance between two elements x and y is taken as $\| x - y \|$. It is a *Hausdorff topological space* [3; 228–229, (A), (B), (C), (5)] if neighborhoods of an element x are spheres consisting of all elements y satisfying $\| x - y \| < \epsilon$ for the positive number ϵ.

Definition 1.1. An *abstract Euclidean space* is a normed linear space such that to each ordered pair of elements x and y there can be associated a number (x, y) with the following properties:

(1) $(tx, y) = t(x, y)$ for all numbers t.
(2) $(x, y) = (y, x)$.
(3) $(x, y + z) = (x, y) + (x, z)$.
(4) $\| x \|^2 = (x, x)$.

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