POWER SERIES WITH INTEGRAL COEFFICIENTS

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1. Introduction. In 1938, Pisot [5] obtained interesting results concerning power series with integral coefficients in connection with the study of the class of algebraic integers whose conjugates lie all inside the unit circle. The numbers of this class (which are obviously real, and can be assumed to be positive) have important properties in the theory of Diophantine approximation, which have been investigated independently by Vijayaraghavan [13], [14]. In 1942–1943 the author [9], [10] has shown that these numbers play a fundamental part in the classification of perfect sets of constant ratio of dissection in sets of uniqueness and sets of multiplicity for trigonometric series. He also proved [11] that the algebraic integers whose conjugates lie all inside the unit circle have the remarkable property of forming a closed set. C. L. Siegel [12] has succeeded in determining the two smallest numbers of the set, which are both isolated. In what follows, these algebraic integers will be called "Pisot-Vijayaraghavan numbers" or briefly "P. V. numbers". (It is convenient to exclude the number 1 from the class of P. V. numbers.)

The purpose of the present paper is to give a new proof of the closure of the set of P. V. numbers, to establish certain theorems on the behavior of power series with integral coefficients, and to give some properties of another class of algebraic integers, namely, the algebraic integers whose conjugates lie all inside or on the unit circle. Apart from the trivial exception of the roots of unity, such algebraic integers are real, of even degree, greater than 1 in absolute value, have actually one conjugate, real, inside the unit circle, and all other conjugates, imaginary, on the unit circle. There is no loss of generality in assuming that such algebraic integers are positive. Numbers of this type and of the fourth degree have already been considered by Vijayaraghavan [13] for problems of Diophantine approximation.

It will be convenient to summarize first the main results of Pisot.

2. Summary of some of Pisot's results.

Theorem A. *Let θ be a real number greater than* 1. *The necessary and sufficient condition in order that a real number $\lambda \neq 0$ should exist such that the series $\sum_0^\infty \sin^2 \pi\lambda\theta^n$ be convergent, is that θ be a P. V. number. Then λ is algebraic and belongs to the field $K(\theta)$.*

See Pisot [5] in which a more general theorem is proved. The proof for the particular Theorem A is reproduced in [9; 222–224].

Theorem A may be stated in another form, which is very different only in appearance. It will be convenient for this purpose, to give a trivial extension

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