INFLUENCE OF THE SIGNS OF THE DERIVATIVES OF A FUNCTION ON ITS ANALYTIC CHARACTER

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1. **Introduction.** In what follows, $f(x)$ denotes a real-valued function defined and of class C^∞ in $[-1, 1]$, i.e., possessing derivatives of all orders in the closed interval $-1 \leqq x \leqq 1$.[1]

Serge Bernstein investigated the analytic nature of functions whose derivatives are each of constant sign in $[-1, 1]$. That such a function is necessarily analytic in $(-1, 1)$ is contained as a very special case in one of his earlier theorems.[2] But he observed also that the signs of the derivatives have a certain influence.[3] If no derivative of $f(x)$ vanishes in $(-1, 1)$, $|f^{(n)}(x)|$ is either steadily increasing or steadily decreasing; we have the first or the second case according as $f^{(n)}(x)f^{(n+1)}(x) > 0$ or $f^{(n)}(x)f^{(n+1)}(x) < 0$ in $(-1, 1)$ (consider the derivative of $[f^{(n)}(x)]^2$). We say that $f^{(m)}(x)$ and $f^{(n)}(x)$ (where $m < n$) belong to the same "block" if $|f^{(m)}(x)|, |f^{(m+1)}(x)|, \cdots, |f^{(n-1)}(x)|, |f^{(n)}(x)|$ all vary in the same sense, i.e., all increase or all decrease. Thus, $f^{(n)}(x)$ and $f^{(n+1)}(x)$ belong to different blocks if and only if

$$f^{(n)}(x)f^{(n+2)}(x) < 0.$$

Let $\lambda_1, \lambda_2, \lambda_3, \cdots$ denote the lengths of the successive blocks into which the sequence $f(x), f'(x), f''(x), \cdots$ is decomposed; we assume here that no block has infinite length, and that, therefore, there is an infinity of blocks. Bernstein found the remarkable result that the lengths of the blocks influence the analytic nature of the function. Roughly stated, the analytic nature of $f(x)$ is simpler if the blocks are shorter. E.g., if the sequence $\lambda_1, \lambda_2, \cdots$ is bounded, $f(x)$ is (or, more precisely, coincides in $[-1, 1]$ with) an entire function of exponential type, i.e., an entire function whose growth does not exceed order one and finite type.

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[1] We write $[a, b]$ for the closed interval $a \leqq x \leqq b$, and (a, b) for the open interval $a < x < b$. The conventions about $f(x)$ do not apply to section 3.

[2] S. Bernstein, (a) *Sur la définition et les propriétés des fonctions analytiques d'une variable réelle*, Mathematische Annalen, vol. 75(1914), pp. 449–468, (b) *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*, Paris, 1926; see especially pp. 196–197. Another proof has been given by R. P. Boas, *Functions with positive derivatives*, this Journal, vol. 8(1941), pp. 163–172.

[3] S. Bernstein, (a) *On certain properties of regularly monotonic functions* (in Russian), Soobshcheniya Kharkovskogo Matematicheskogo Obshchestva (Communications de la Société Mathématique de Kharkow), (4), vol. 2(1928), pp. 1–11, (b) *Sur les fonctions régulièrement monotones*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, vol. 186(1928), pp. 1266–1269, (c) Same title, Atti del Congresso Internazionale dei Matematici, Bologna, 1928, vol. 2(1930), pp. 267–275.

Observe that since $f(x)$ is analytic in $(-1, 1)$ and not a polynomial, if no derivative changes sign in $(-1, 1)$, then no derivative can vanish there.