THE DIVERGENCE OF NON-HARMONIC GAP SERIES

By Philip Hartman

It was recently shown[1] that if $\lambda_0, \lambda_1, \cdots$ is a sequence of positive real numbers satisfying the gap condition

$$(1) \qquad \frac{\lambda_k}{\lambda_{k-1}} > q > 1 \qquad (k = 1, 2, \cdots),$$

then the convergence of the series

$$(2) \qquad \sum_{k=0}^{\infty} | a_k |^2$$

implies the convergence of

$$(3) \qquad \sum_{k=0}^{\infty} a_k e^{i\lambda_k t}$$

for almost all t, $-\infty < t < +\infty$, while if (1) is modified so that "$q > 1$" is replaced by "$q > \frac{1}{2}(5^{\frac{1}{2}} + 1)$", then the divergence of (2) implies the divergence of (3) for almost all t, $-\infty < t < +\infty$. The object of this note is to show that *the condition* (1), without any modification, *and the divergence of* (2) *implies the divergence of the series* (3) *for almost all* t, $-\infty < t < +\infty$.

In order to prove this statement, let $\sigma(E)$ denote the completely additive measure on the t-axis which has the non-negative density[2]

$$(4) \qquad \frac{(1 - \cos t)}{\pi t^2} \qquad (-\infty < t < +\infty)$$

so that, if E is a measurable set,

$$(5) \qquad \sigma(E) = \int_E \frac{1 - \cos t}{\pi t^2} \, dt.$$

Obviously, for any measurable set E,

$$(6) \qquad 0 \leqq \sigma(E) \leqq 1.$$

The Fourier-Stieltjes transform of this σ-measure,

$$(7) \qquad \int_{-\infty}^{+\infty} e^{i\lambda t} \, d\sigma(t) = \max (1 - | \lambda |, 0),$$

vanishes for all $| \lambda | \geqq 1$.

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[1] M. Kac, *Convergence and divergence of non-harmonic gap series*, Duke Mathematical Journal, vol. 8(1941), pp. 541–545.

[2] The introduction of this measure function avoids the awkward construction, used in loc. cit., see footnote 1, of a measure whose Fourier-Stieltjes transform vanishes on a particular sequence of points. It also makes it possible to use, with only the slightest of modifications, the method applied by A. Zygmund, *Trigonometrical Series*, Warsaw, 1935, pp. 120–122, in the case that the frequencies λ_k are integers.