A GENERAL EQUATION FOR RELAXATION OSCILLATIONS

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1. **Introduction.** The importance of relaxation oscillations in physical and engineering problems was shown by Van der Pol,[1] who also treated by graphical methods a particular equation describing these oscillations. The origin of relaxation oscillations can be described *qualitatively* by considering the following two equations with constant coefficients:

$$\ddot{x} + 2a\dot{x} + bx = 0, \tag{1.1}$$

$$\ddot{x} - 2a\dot{x} + bx = 0, \tag{1.2}$$

where $a > 0$, $b > a^2$, and the differentiations are with respect to t. If we denote $b - a^2$ by ω^2 then the solution of (1.1) is $Ae^{-at} \sin (\omega t + \alpha)$ and the solution of (1.2) is $Ae^{at} \sin (\omega t + \alpha)$, where A and α are arbitrary constants. In an electrical circuit, for example, described by (1.1), the term $2a\dot{x}$ arises from a withdrawal of energy from the system. Since there is no energy being put into the system, this withdrawal is uncompensated and results in a gradual dissipation of the initial energy of the system. Thus the solution of (1.1) tends to zero. In (1.2), on the other hand, the only term affecting the energy is $-2a\dot{x}$ which arises from adding energy to the system. Thus the solution of (1.2) describes oscillations of every increasing amplitude.

The equation (1.1) may be said to describe a system with positive damping and (1.2), a system with negative damping. Positive damping decreases the energy and therefore the amplitude of an oscillation while negative damping increases it. Relaxation oscillations arise when both positive and negative damping occur in a system. More precisely what occurs is that for small displacements, that is, when x is small, the system has negative damping which causes oscillations of increasing amplitude; on the other hand, for large displacements the system has positive damping which tends to decrease the amplitude of oscillation. Clearly, then, the steady state amplitude of oscillation of the system cannot be too small, for then damping would always be negative and the oscillation would increase in amplitude, that is, would not be steady-state. In the same way a very large amplitude is not possible. Thus qualitatively one would expect a steady-state oscillation of such amplitude that during each period the energy lost when the displacement was large (and damping positive) would be exactly compensated by the energy gained when the displacement was small (and damping negative).

All this is what one would expect qualitatively. Actually, under a wide range of conditions precisely this does happen. On the other hand, there are also

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[1] B. Van der Pol, *Relaxation-Oscillations*, Philosophical Magazine, seventh series, vol. 2(1926), p. 978.