SOME PROPERTIES OF SUMMABILITY

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Introduction. The purpose of this note is to present a few general results pertaining to some well-known and desirable properties of summability. By a method of summability, we shall ordinarily mean the familiar matrix method of assigning a limit to a sequence, although some of the broader remarks apply to any method of summability. The principal results are derived for the class of *reversible*[1] matrix methods, namely those for which the system of equations

$$\sum_{k=1}^{\infty} a_{mk}\, s_k = t_m \qquad (m = 1, 2, 3, \cdots)$$

has a unique solution $\{s_k\}$ corresponding to each convergent sequence $\{t_m\}$. The complex number system is employed throughout except where the contrary is specified.

1. **Translative methods.** We shall say that a method of summability is *translative to the right* (*to the left*) if the summability of the sequence $s_1, s_2, s_3, \cdots$ to the limit s always implies the summability of the sequence $s_2, s_3, s_4, \cdots$ ($s_0, s_1, s_2, \cdots$, for arbitrary s_0) to the limit s. A method that is both translative to the right and to the left will be called simply *translative*. We shall concern ourselves principally with methods of this latter type. It is immediately obvious that the summability of $s_1, s_2, s_3, \cdots$ to s by means of a translative method implies the summability of $s_{m+1}, s_{m+2}, s_{m+3}, \cdots$ to s, where m may be any positive or negative integer and s_k is to be interpreted as arbitrary if $k \leqq 0$. Furthermore, if the sequence of partial sums of the series $u_1 + u_2 + u_3 + \cdots$ is summable to s by means of a linear and regular translative method, then the sequences of partial sums of the series $u_2 + u_3 + u_4 + \cdots$ and $u_0 + u_1 + u_2 + \cdots$ (u_0 arbitrary) are summable to $s - u_1$ and $s + u_0$, respectively. Conversely, if the summability of the series $u_1 + u_2 + u_3 + \cdots$ to s by means of a linear regular method A always implies the simultaneous summability of the series $u_2 + u_3 + u_4 + \cdots$ and $u_0 + u_1 + u_2 + \cdots$ to $s - u_1$ and $s + u_0$, respectively, it follows that A must be translative.

The property described here as translativity has been mentioned by several writers[2] as a desirable adjunct to a method of summability, although the known

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[1] See S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 90. The term *reversible* has been used by I. Schur (*Über die Äquivalenz der Cesàroschen und Hölderschen Mittelwerte*, Mathematische Annalen, vol. 74(1913), pp. 447–458) in a sense different from that of Banach.

[2] The reader will find references easy to locate in L. L. Smail, *History and Synopsis of the Theory of Summable Infinite Processes*, University of Oregon Press, 1925.