THE CONTINUED FRACTION AS A SEQUENCE OF LINEAR TRANSFORMATIONS

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1. **Introduction.** This paper contains a development of properties of the continued fraction

$$\frac{1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\frac{a_4}{1}+\cdots \tag{1.1}$$

in which the elements $a_2, a_3, a_4, \cdots$ are complex numbers. The central idea may be described as follows. With the continued fraction (1.1) we associate a sequence of linear transformations[1]

$$a_1(v) = v, \qquad a_k(v) = \frac{1}{1 + a_k v} \qquad (k = 2, 3, 4, \cdots). \tag{1.2}$$

Then the product of the first n of these is

$$a_1 a_2 \cdots a_n(v) = \frac{1}{1}+\frac{a_2}{1}+\cdots+\frac{a_{n-1}}{1}+\frac{a_n v}{1} = \frac{a_n A_{n-2} v + A_{n-1}}{a_n B_{n-2} v + B_{n-1}}, \tag{1.3}$$

where A_k and B_k are the k-th numerator and denominator of (1.1), i.e., $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = 0$, $B_0 = 1$, $A_k = A_{k-1} + a_k A_{k-2}$, $B_k = B_{k-1} + a_k B_{k-2}$ ($k = 1, 2, 3, \cdots$; $a_1 = 1$). Corresponding to an arbitrary set V of points in the complex z-plane, called a *value region*, we determine a set $\mathfrak{A}$ of points, called the *element region* (corresponding to V), by the condition that a is in $\mathfrak{A}$ if and only if the transformation $w = a(v) = 1/(1 + av)$ transforms V into a subset of itself, i.e., $a(V) \subset V$. It is at once evident that if $a_2, a_3, a_4, \cdots$ are in $\mathfrak{A}$, and

$$a_1 a_2 \cdots a_n(V) = V^{(n)} \qquad (n = 1, 2, 3, \cdots),$$

then

$$V = V^{(1)} \supset V^{(2)} \supset V^{(3)} \supset \cdots.$$

Hence, if V is a bounded closed set so that $V^{(1)}, V^{(2)}, V^{(3)}, \cdots$ are closed, then there are just two cases, namely:

Case I. The sets $V^{(n)}$ ($n = 1, 2, 3, \cdots$) have one and only one point, v_0, in common.

Case II. The sets $V^{(n)}$ ($n = 1, 2, 3, \cdots$) have two or more points in common.

In Case I, we have, *uniformly* for all v in V:

$$\lim_{n=\infty} a_1 a_2 \cdots a_n(v) = v_0.$$

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[1] It is convenient to use the symbol a_k in two senses. The subscript 1 will be reserved for the identity transformation.