CENTRAL CHAINS OF IDEALS IN AN ASSOCIATIVE RING

BY S. A. JENNINGS

In this paper we have two main ideas in mind. It is well known that there is a considerable similarity between those properties of a group associated with its commutator structure and the theory of Lie rings.[1] We show first that, with suitable definitions of "commutator ideal", many of the properties of commutator subgroups have analogues in the theory of associative rings. In particular, we are interested in extending the notions of "nilpotent group" and "solvable group" to rings. On the other hand, every associative ring determines a Lie ring, which we have called the "associated Lie ring" (cf. §6), and the question arises as to how far the solvability or nilpotency of this Lie ring determines corresponding properties of the original associative ring as we have defined them in the first part. In the case of algebras we answer this question completely and show that if the Lie algebra is solvable (or nilpotent) the associative algebra has the corresponding property. For a general ring, however, we obtain only a partial answer.

In seeking an analogue for the commutator subgroup of two given normal subgroups, a difficulty arises at once. As is well known, the subgroup generated by all commutators of a group is a normal subgroup; in a ring the subring generated by all elements of the form $xy - yx$ is not in general an ideal. We overcome this difficulty by defining the "commutator ideal" of two given ideals A, B of the associative ring R as the smallest ideal of R containing all elements of the form $ab - ba$, where $a \in A, b \in B$. However, there are some disadvantages to this definition, as compared with the corresponding one in the theory of groups, and in consequence, commutator subgroups enjoy some properties which have no analogues for commutator ideals. In a group, a normal subgroup, besides having the property that its residue classes again form a group, is such that it is transformed into itself by inner automorphisms. Ideals in a ring play no such dual rôle, and in general those properties of commutator subgroups depending on the second of these facts have no analogue in our theory.

1. **Commutators.** Let A and B be any two (not necessarily distinct) ideals of an associative ring R. We define the *commutator ideal*, $A \circ B$, of A and B to be the ideal of R generated by all elements of the form $ab - ba$, where $a \in A$, $b \in B$, that is, $A \circ B$ is the smallest ideal of R containing all elements of the form $ab - ba$.

In what follows, it will be convenient to write $xy - yx$ as $x \circ y$; to avoid the possibility of confusion between this notation and that defined above for com-

Received November 8, 1941. Presented to the American Mathematical Society, September 2, 1941.

[1] For properties of commutator subgroups see [1] and [7]. (Numbers in brackets refer to the bibliography.) The relations between groups and Lie rings are discussed in [5] and [6].