ALGEBRAIC PROPERTIES OF CERTAIN MATRICES OVER A RING

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1. **Introduction.** A considerable part of recent work in the theory of matrices has been devoted to the study of matrices with elements in some domain more general than the field of complex numbers, which is so prominent in early papers on this subject. By restricting the domain to be a suitably chosen field or ring, different parts of the classical theory have been generalized, or analogues found, in various ways. In the case in which the elements are from a non-commutative ring, two quite different approaches to the subject have been used. In one of these, there is no attempt to introduce the concept of *determinant*, but sufficiently strong divisibility conditions are assumed in order to carry over certain parts of the theory. In the other approach, which is used in this paper, the class of matrices considered is restricted in such a way that determinants, having many of the familiar properties of ordinary determinants, can be defined for the matrices under consideration. Reference here can be made to the work of E. H. Moore ([6],[1] Chap. II) on Hermitian matrices in what he calls a "number system of type B". As the present investigation was inspired by this work of Moore or, more precisely, by the similarity between his theorems and known theorems about arbitrary matrices over a commutative ring, we pause to describe briefly the class of matrices to which Moore's theory is directly applicable.

For the moment, let $\mathfrak{T}$ be a ring with unit element 1, in which the equation $2x = 1$ has a unique solution and in which there is defined an anti-automorphism or *involution* $a \to \bar{a}$. Thus

$$\overline{a + b} = \bar{a} + \bar{b}, \qquad \overline{ab} = \bar{b}\bar{a}, \qquad \bar{\bar{a}} = a.$$

We require also that the elements a of $\mathfrak{T}$ such that $a = \bar{a}$, the so-called *symmetric elements*, shall be in the center of $\mathfrak{T}$. Such a ring may be called an *involutorial ring*. If $A = (a_{ij})$ is a square matrix with elements in $\mathfrak{T}$, and $a_{ij} = \bar{a}_{ji}$, then A is said to be a *Hermitian matrix*. Now a "number system of type B", as defined by Moore, is a special instance of an involutorial ring and is in fact either a commutative field, of characteristic other than 2, with $a = \bar{a}$, or a quadratic field over the field of symmetric elements or a generalized quaternion algebra over this field. However, Jacobson has pointed out in [2] that Moore's definition of determinant and many of his results remain valid for Hermitian matrices over any involutorial ring.

Although many of Moore's theorems coincide in statement with known theorems about matrices over a commutative ring, the published proofs are quite different. We shall begin by showing how to unify these two cases, at least to a

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.