THE STRUCTURE OF THE GROUP OF $\mathfrak{P}$-ADIC 1-UNITS

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Introduction. It is well known that in local $\mathfrak{P}$-adic fields the logarithm function can be defined by the series

$$-\log (1 - x) = \sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu},$$

where x is a $\mathfrak{P}$-adic number, the series converging for all x with $| x |_{\mathfrak{P}} < 1$. Although the logarithm is needed in various connections for algebraic number theory, very little is known about it. Even its value domain is still unknown, except for this fact: $\log (1 - x)$ maps the set x for which $| x |_{\mathfrak{P}} < | p |^{\frac{1}{p-1}}$ onto itself in a one-to-one way. However, the mapping of those x for which $| p |^{\frac{1}{p-1}} \leqq | x |_{\mathfrak{P}} < 1$ is still unknown, and it is this which must be investigated.

Many of the explicit formulas for the reciprocity law in algebraic number fields are best stated by means of the $\mathfrak{P}$-adic logarithm.[1] Although these explicit formulas have been proved, they are not clearly understood; it is probable that complete knowledge of the value domain of the $\mathfrak{P}$-adic logarithm would better our understanding of the formulas. This knowledge would be of use also for other applications to algebraic number theory.

Let K be a $\mathfrak{P}$-adic number field; then all units ϵ which are congruent to 1 modulo $\mathfrak{P}$—the 1-units of Hensel—constitute the set of elements in K having logarithms. If the structure of this multiplicative group of 1-units were completely known in some convenient way, then also the value domain of the $\mathfrak{P}$-adic logarithm would be known. M. Krasner has attacked this problem,[2] considering the general case where K is normal over a field k. His method was the following. Let G be the Galois group of K/k, σ, τ, $\cdots$, its elements, and form the group ring Γ of G taken over the ring of p-adic integers; Γ consists of elements $\zeta = a\sigma + b\tau + \cdots$, where a, b, $\cdots$ are p-adic integers. If ϵ is a 1-unit, then the hypercomplex power ϵ^ζ can be defined in the usual way,

$$\epsilon^\zeta = (\sigma\epsilon)^a(\tau\epsilon)^b \cdots = \sigma(\epsilon^a)\tau(\epsilon^b) \cdots$$

with $(\epsilon^{\zeta_1})^{\zeta_2} = \epsilon^{\zeta_2\zeta_1}$. In this way, Γ is seen to be a ring of operators on the group of 1-units. Krasner tried to find a minimal basis for the 1-units, taking the hypercomplex exponents Γ as operator domain; that is, he tried to find the fewest number of 1-units ϵ_1, ϵ_2, $\cdots$, ϵ_r, such that $\epsilon_1^\Gamma \epsilon_2^\Gamma \cdots \epsilon_r^\Gamma$ give all 1-units in K (except perhaps for roots of unity). It was hoped that an independent

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[1] See [2] (numbers in brackets refer to the bibliography); see chapter IV on explicit formulas for the reciprocity law and the $\mathfrak{P}$-adic logarithm.

[2] See M. Krasner [5]. All references to Krasner have to do with this paper.