ADDITIVE FUNCTIONS AND ALMOST PERIODICITY

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1. Let $f = f(n)$ be a function defined for $n = 1, 2, \cdots$. Its mean-value, $M(f)$, is usually defined by

$$\frac{f(1) + \cdots + f(n)}{n} \to M(f) \quad \text{as} \quad n \to \infty,$$

provided that this limit exists (this proviso should include that $M(f) \neq \pm\infty$). However, in certain connections, this definition of a mean-value turns out to be too vague. Correspondingly, the *Disquisitiones Arithmeticae* consider a more restrictive definition,[1] which today can be formulated as follows:

$$\frac{f(m+1) + \cdots + f(m+n)}{n} \to M(f) \quad \text{as} \quad n \to \infty$$

uniformly for all $m(= 1, 2, \cdots)$. Let $M(f)$ then be denoted also by $M^*(f)$.

Thus $M(f)$ exists and equals $M^*(f)$ whenever $M^*(f)$ exists. But $M^*(f)$ need not exist when $M(f)$ exists. The situation is illustrated by the following observations:

(i) If $f(n) = 1$ or $f(n) = n^{\frac{1}{2}}$ according as n is not or is a perfect square, then $M(f)$ exists. Hence $\limsup f(n) = \infty$ does not preclude the existence of $M(f)$ for an f. But this situation is changed if M is replaced by M^*, since $|f(n)| <$ const. is a necessary condition for the existence of $M^*(f)$. In fact, the existence of $M^*(f)$ means that, if $\epsilon > 0$ is arbitrary and if N_ϵ is suitably chosen, then

$$\left| \sum_{k=m}^{m+n-1} f(k) - nM^*(f) \right| < \epsilon n \quad \text{whenever} \quad n \geqq N_\epsilon,$$

where m is arbitrary. Hence, if $\epsilon = 1$ and $N = N_1$, then

$$\left| \sum_{k=m}^{m+N} f(k) - (N+1)M^*(f) \right| < N + 1, \qquad \left| \sum_{k=m+1}^{m+N} f(k) - NM^*(f) \right| < N$$

for every m. Consequently, for every m,

$$|f(m) - M^*(f)| < N + 1 + N.$$

Since $M^*(f)$ and N are independent of m, it follows that f is bounded.

(ii) Birkhoff's ergodic theorem states that, under his assumptions, the sequence of images of an arbitrary L-integrable function possesses an M-mean almost everywhere. But the theorem becomes false if M is replaced by M^*. This is clear from (i) since the L-integrable function can be chosen rather un-

Received September 30, 1941.

[1] C. F. Gauss, *Werke*, vol. 1, pp. 362–366.