THE COEFFICIENTS OF THE RECIPROCAL OF A SERIES

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1. **Introduction.** Consider the elliptic function $\wp(u)$ with invariants $g_2 = 4$, $g_3 = 0$, so that $\wp(u)$ satisfies the differential equation

$$\wp'^2(u) = 4\wp^3(u) - 4\wp(u). \tag{1.1}$$

Put

$$\wp(u) = \frac{1}{u^2} + \sum_{m=1}^{\infty} \frac{2^{4m}E_m}{4m} \frac{u^{4m-2}}{(4m-2)!}, \tag{1.2}$$

where E_m are rational. Then Hurwitz[1] has proved the following theorem:

$$E_m = G_m + \frac{1}{2} + \sum \frac{(2a)^{\frac{4m}{p-1}}}{p}, \tag{1.3}$$

where G_m is integral and the summation is extended over those primes $p = 4k + 1$ such that $p - 1 \mid 4m$; furthermore the odd integer a is determined by means of

$$p = a^2 + b^2, \qquad a \equiv b + 1 \pmod 4.$$

The method of proof depends in particular on the complex multiplication of $\wp(u)$, and Hurwitz suggests that like theorems may hold for the coefficients of those elliptic functions that possess complex multiplication. For the case in which the ratio of the periods is an imaginary cube root of unity, this was indeed proved by Matter.[2]

In the present paper we consider the class of series

$$f(u) = \sum_{1}^{\infty} \frac{c_m u^m}{m!} \qquad (c_1 = 1), \tag{1.4}$$

where the c_m are integral, and assume that the inverse of $f(u)$ has the form

$$\lambda(u) = \sum_{1}^{\infty} \frac{\epsilon_m u^m}{m},$$

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[1] A. Hurwitz, *Über die Entwickelungscoeffizienten der lemniscatischen Functionen*, Mathematische Annalen, vol. 51(1899), pp. 196–226 = *Mathematische Werke*, Basel, 1933, vol. II, pp. 342–373.

[2] K. Matter, *Die den Bernoulli'schen Zahlen analogen Zahlen im Körper der dritten Einheitswurzeln*, Zürich, 1900; reviewed in Jahrbuch der Fortschritte der Math., vol. 31(1900), pp. 204–206.