THE EXTENSION OF RECTANGLE FUNCTIONS

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Introduction

0.1. The theory of Lebesgue measure extends in a completely additive way the concept of area from a class of simple figures—circles, triangles, rectangles, and polygons in general—to a more comprehensive class of sets which is closed under the operations of addition, subtraction, multiplication, and limit. It has inspired many papers and books[1] dealing with the problem of securing a completely additive extension[2] of a general set function defined on a certain class of simple figures. For most applications, however, each of these general theories possesses certain artificial features, some of which we shall mention later (cf. §0.3). The purpose of this paper is to set forth a simple and natural theory which meets the needs of many applications. We shall restrict our considerations to the plane, since it at once offers rather "heavy" sets which may carry no weight and rather "meager" sets which may carry considerable weight. Before stating our results, we shall summarize certain important results in the literature. Results of Radon (cf. §0.2) will be carefully summarized since they will be used in establishing our theorems. Results of Caccioppoli (cf. §0.4) will also be stated because, while they would be quite useful in the applications, unfortunately they are false; we shall give a counter example to show this in §0.4.

0.2. Let R_0 be a fixed rectangle in the xy-plane bounded by the lines $x = 0$, $x = x_0 > 0; y = 0, y = y_0 > 0$.[3] A class of sets in R_0 is said to be *closed* (relative to R_0)[4]—and is denoted generically by K—if the following conditions are satisfied:

(i) every open set (relative to R_0) is in K;
(ii) if e is a set in K, then $R_0 - e$ is also a set in K;

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[1] See the bibliography at the end of this paper. Numbers in square brackets refer to references in this bibliography.

[2] For definitions of such concepts as set functions, completely additive set functions, completely additive extensions, etc., see, for example, [3], [4], [6].

[3] Rectangles, as R_0, whose sides are parallel by pairs to the x- and y-axes respectively are termed *oriented* rectangles.

[4] Radon (cf. [3]) and de la Vallée-Poussin (cf. [6]) define "Classe T" and "corps fermé" respectively; for sets in R_0, each of these concepts is equivalent to our closed class.