EXTENSION OF THE RANGE OF A DIFFERENTIABLE FUNCTION

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1. **Introduction.** In 1934 Whitney[1] showed that a function $f(x) = f(x_1, \cdots, x_n)$ of class C^m on a closed set A in Euclidean n-space E can be extended so as to be of class C^m on the whole space E. In fact he proved that $f(x)$ can be extended so as to be of class C^∞ on $E - A$. Using this result he showed further that the extension can be made so that $f(x)$ is analytic on $E - A$. In the present paper two methods of extending the range of differentiable functions are given. The first method (given in §3 below) is applicable only when m is finite and the boundary of A has suitable properties. It is, however, sufficiently general to be of interest, and the proof is relatively simple. The method used is a generalization of the reflection principle used by L. Lichtenstein[2] when $n = 3$ and $m = 1$. The second method (given in §§4 and 5 below) is essentially a modification of the one given by Whitney and is applicable to functions of class C^m (m finite or infinite) on an arbitrary closed set A. The details of the proof appear to be simpler than those of Whitney's. The extension is of class C^∞ on $E - A$ in this case.

2. **Notations and definitions.** In the following pages we shall use essentially the notations and terminology used by Whitney.[3] An n-tuple $k_1, \cdots, k_n$ of non-negative integers will be denoted by a single symbol k, and we write

$$k! = k_1!k_2! \cdots k_n!, \qquad \sigma_k = k_1 + \cdots + k_n, \qquad f_k(x) = f_{k_1\cdots k_n}(x)$$

$$f_0(x) = f_{0\cdots 0}(x), \qquad D_0 f(x) = f(x), \qquad D_k f(x) = \frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f(x).$$

By the symbol

$$P_m(x, x') = \sum_k \frac{f_k(x')}{k!} (x - x')^k \qquad (\sigma_k \leqq m) \tag{1}$$

will be meant the sum of all terms of the form

$$\frac{f_{k_1\cdots k_n}(x')}{k_1! \cdots k_n!} (x_1 - x_1')^{k_1} \cdots (x_n - x_n')^{k_n}$$

for which $\sigma_k \leqq m$. We set

$$P_{m;k}(x, x') = D_k P_m(x, x'), \tag{2}$$

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[1] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society, vol. 36(1934), pp. 63–89.

[2] L. Lichtenstein, *Eine elementare Bemerkung zur reellen Analysis*, Mathematische Zeitschrift, vol. 30(1929), pp. 794–795.

[3] Loc. cit., p. 64.