OPERATION THEORY AND MULTIPLE SEQUENCE TRANSFORMATIONS

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Consider the problem: To derive sets of conditions on the $(l + n)$-dimensional matrix of complex numbers $(a_{m^{(1)},m^{(2)},\ldots,m^{(l)};k^{(1)},k^{(2)},\ldots,k^{(n)}})$ $(l + n \geqq 2)$ necessary and sufficient that the l-tuple sequence $\{\sigma_{m^{(1)},m^{(2)},\ldots,m^{(l)}}\}$ belong to a prescribed class whenever the n-tuple sequence of complex numbers $\{s_{k^{(1)},k^{(2)},\ldots,k^{(n)}}\}$ belongs to a prescribed class, where

$$\sigma_{m^{(1)},m^{(2)},\ldots,m^{(l)}} = \sum_{k^{(1)},k^{(2)},\ldots,k^{(n)}=1}^{\infty} a_{m^{(1)},m^{(2)},\ldots,m^{(l)};k^{(1)},k^{(2)},\ldots,k^{(n)}}\, s_{k^{(1)},k^{(2)},\ldots,k^{(n)}} .$$

This problem was solved in[1] H_1 and H_2 for each of the 256 cases corresponding to 16 classes of multiple sequences ranging from that in which the sequence is convergent and all partial limits exist and are zero to that in which the elements of the sequence are merely bounded for all values of the subscripts which are sufficiently large.

However, these derivations were based exclusively on "classical" methods, and this fact left open for a time the question of applicability of operation methods to multiple sequence transformations. The authors have now succeeded in applying these methods in the cases treated in H_1 and H_2 and our present purpose is to exhibit such phases of this application as may conceivably be of value in future investigations of multiple sequence transformations.

In case $l = n = 1$ we have to deal with the matrix (a_{mk}) and sequences $\{s_k\}$ and $\{\sigma_m\}$ related by the equation $\sigma_m = \sum_{k=1}^{\infty} a_{mk}s_k$. The classes of sequences which seem to have been of interest here are those of null, convergent, and bounded sequences.

We may remark parenthetically that the determination of "regularity" conditions on (a_{mk}), that is, conditions such that σ_m converge to $\lim_{k\to\infty} s_k$ whenever the latter exists, constitutes no separate problem, since we need here merely to determine conditions on the matrix (b_{mk}) necessary and sufficient that $\{\tau_m\}$ be a null sequence whenever $\{s_k\}$ is convergent, where $b_{mk} = a_{mk} - \delta_m^k$ and $\tau_m = \sum_{k=1}^{\infty} b_{mk}s_k$, and δ_m^k is Kronecker's symbol. Similar reduction to homogeneous

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[1] H_1 and H_2 denote the papers by Hamilton, *Transformations of multiple sequences*, this Journal, vol. 2(1936), pp. 29-60, and *Change of dimension in sequence transformations*, ibid., vol. 4(1938), pp. 341–342, respectively.