PSEUDO-NORMED LINEAR SPACES

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Hyers [2][1] introduced the concept of pseudo-normed linear spaces (p.l.s.'s) and showed that such spaces are equivalent to linear topological spaces (l.t.s.'s). Throughout the present paper we shall deal with p.l.s.'s, as it is in general more convenient, though it is to be remembered that what we prove for p.l.s.'s applies also to l.t.s.'s. The terms used which are not defined are those for l.t.s.'s (see [3]). For instance, when we speak of a convex p.l.s., we mean that the l.t.s. that is equivalent to the p.l.s. is convex. In this paper a necessary and sufficient condition is given that there exist a non-null linear functional on a p.l.s., and also it is shown that the set of all linear functions on a p.l.s. to any other p.l.s. is itself a p.l.s. It is then shown by example[2] that there do exist p.l.s.'s on which no non-null linear functional can be defined.

1. Let T with elements $x, y, \cdots$ be a p.l.s. with respect to a strongly partially ordered space [2] D; that is, T is a linear space such that there exists a real-valued function $n(x, d)$ defined on TD which satisfies the following postulates:

(1) $n(x, d) \geqq 0$; $n(x, d) = 0$ for all $d \epsilon D$ implies $x = \theta$, where θ is the zero element of T;

(2) $n(\alpha x, d) = |\alpha| n(x, d)$ for all $x \epsilon T$, $d \epsilon D$, and α real;

(3) given $\eta > 0$, $e \epsilon D$, there exist $\delta > 0$, $d \epsilon D$ such that $n(x + y, e) < \eta$ for $n(x, d) < \delta$ and $n(y, d) < \delta$;

(4) $d > e$ implies that $n(x, d) \geqq n(x, e)$.

$n(x, d)$ is called the pseudo-norm of x with respect to d.

Let αd represent the association of a positive real number α with an element $d \epsilon D$. Define[3] $1 \cdot e = e$, $\alpha(\beta e) = (\alpha\beta)e = (\beta\alpha)e$, $E = [\alpha d; d \epsilon D, \alpha > 0]$ and $n(x, \alpha d) = \alpha n(x, d)$, for $\alpha, \beta > 0$. Then E is a strongly partially ordered space with $e_1 \geqq e_2$, $e_1, e_2 \epsilon E$, if $n(x, e_1) \geqq n(x, e_2)$ for all $x \epsilon T$, and $e_1 = e_2$ if $n(x, e_1) = n(x, e_2)$ for all $x \epsilon T$. This is consistent with the definitions already given. $n(x, e)$ is a pseudo-norm of x with respect to e; that is, $n(x, e)$ satisfies postulates (1)–(4). This modified pseudo-norm gives a more convenient statement of (3), namely,

(3′) given $e \epsilon E$ there exist $f \epsilon E$ such that $n(x + y, e) \leqq n(x, f) + n(y, f)$ for all $x, y \epsilon T$;

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[1] The numbers in brackets refer to the bibliography at the end of the paper.

[2] The author is indebted to the referee for this example. For another example see Theorem 1 of [1]. A proof of Theorem 1 of [1] can be given if we use Theorem 4 of this paper.

[3] $[x;\ ]$ denotes the set of all x's having the property following the semicolon.