REAL INVERSION FORMULAS FOR LAPLACE INTEGRALS

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1. **Introduction.** In this paper we shall be concerned with real inversion formulas for the Laplace integrals

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t), \tag{1}$$

$$f(x) = \int_0^\infty e^{-xt}\phi(t)\, dt. \tag{2}$$

In (1) $\alpha(t)$ is of bounded variation in $(0, R)$ for all positive R, and normalized, that is,

$$\alpha(0) = 0; \qquad \alpha(t) = \tfrac{1}{2}[\alpha(t+) + \alpha(t-)] \qquad (t > 0).$$

In (2) $\phi(t)$ is assumed Lebesgue integrable in $(0, R)$ for all positive R.

We shall obtain new inversion formulas for (1) and (2) in terms of

I. $f(x)$ and its derivatives in a neighborhood of infinity,

II. $f(x)$ and its differences in a neighborhood of infinity,

III. $f(x)$ and its derivatives on a discrete set of points near infinity.

In the special case of the integral (2) it will suffice to know the derivatives or differences of high enough order.

Case I has been treated by several writers, particularly D. V. Widder. Theorem 1.1, which forms the basis of our work, is an extension of results due to Widder and the present author [2].[1]

Case II can be handled by means of an operator employed by Widder to solve the Hausdorff moment problem ([6], pp. 174 and 178). The result is that if (2) converges, then[2]

$$\phi(t) = \lim_{k\to\infty} \frac{(n+k+1)!}{n!\,k!}(-1)^k \Delta_1^k f(n+1), \qquad n = \left[\frac{k}{e^t - 1}\right],$$

for almost all $t > 0$. There is a similar result for the integral (1). Our formulas differ somewhat from these.

Case III does not appear to have been treated previously.[3]

The main theorem, which we now state, concerns the integral (2). D will

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

[2] As usual, $[a]$ denotes the largest integer in a.

[3] For a summary of methods of inversion when other information concerning $f(x)$ is given see [1], p. 1.