REGULAR CURVE-FAMILIES FILLING THE PLANE, I

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Introduction. In studying the solutions of differential equations $dx/dt = f(x, y)$, $dy/dt = g(x, y)$ one is led to the concept of a *family of curves* in the plane.[1] Also the level-curves of a function $h(x, y)$ form a family of curves in the plane. In both cases, under proper restriction on the functions involved, the following condition, which we term *regularity*, holds: *the family is locally homeomorphic with parallel lines.*

It is the object of the present paper to consider the properties of a curve-family in the plane, given only that it satisfies the regularity condition in an open subset R of the plane. The principal theorem of the paper (see §4.2, Theorem 42 below) is that, *when R is the whole plane, the given family is a level-curve family of some function $h(x, y)$.* This generalizes a theorem of Kamke.[2]

In order to obtain this result the structural features of such a curve-family will be analyzed in some detail. In a paper to follow the structural question will be more completely considered and a topological classification of all families regular in the entire plane will be obtained.

In the first part of the paper the results of Bendixson on families defined by differential equations will be generalized to the more general (topologically defined) families. The purpose of the generalization is partly to show that Bendixson's theorems follow almost completely from the regularity alone, differentiability being inessential, and partly to express those theorems in the topological language which their nature demands. This is done particularly in the case of the discussion of the neighborhood of a closed curve (see §1.8 below). Other theorems considered are that a closed curve must enclose a singularity (see Theorem 13) and that an open curve which in one direction remains bounded and has no singularity as limit is asymptotic to a closed curve (see Theorem 11 below).

In the second part the structure of the families filling the plane will be analyzed and formulated in an algebraic manner. For such families each curve is open

Received by the Editors of the Annals of Mathematics, February 21, 1940, accepted by them and later transferred to this Journal. The material in this paper and the one to follow is taken from the author's Ph.D. thesis at Harvard University. The author expresses his gratitude to Professor Hassler Whitney for his advice in the preparation of the paper.

[1] In the bibliography at the end of this article detailed references are given to the classical works of Poincaré and Bendixson on the subject, also to works of Brouwer, v. Kerékjártó, Kneser, Denjoy, Birkhoff, Whitney, and George. Numbers in brackets refer to this bibliography.

[2] See E. Kamke, *Zur Theorie der Differentialgleichungen*, Mathematische Annalen, vol. 99(1928), p. 613.