TWO-TO-ONE TRANSFORMATIONS

By J. H. Roberts

1. An n-to-1 transformation is one for which every inverse image consists of exactly n points. This paper is concerned with *continuous* n-to-1 transformations defined over separable metric spaces. Except in this introductory paragraph only the case $n = 2$ is considered. Now a continuous n-to-1 transformation is in some respects like an at most n-to-1 interior transformation. In particular they are alike in not altering dimension, if n is finite.[1] However, an n-to-1 continuous transformation with $n = \aleph_0$ can increase dimension at will, in contrast to the case of the at most $\aleph_0$-to-1 interior transformation. Transformations which are both n-to-1 and interior have been considered by G. T. Whyburn.[2]

Now O. G. Harrold has shown[3] that no continuous 2-to-1 transformation can be defined on an arc. The main object of the present paper is to prove the following

Principal Theorem. *There does not exist a continuous* 2-*to*-1 *transformation defined on a closed* 2-*cell.*

It seems probable that the same result holds for the closed n-cell, for every n. For this reason certain lemmas have been formulated as more general theorems, with the hope that they may be of use later on.

2. Throughout this paper we use the following notation: T is a continuous, 2-to-1 transformation defined over a space M with differing topological properties, as specified in the various theorems. In every case M is at least metric. No mention is made of the space into which M is mapped, but it can always be taken as a subset of Hilbert space. The set of inverse images is an upper semi-continuous collection G filling M, and every element of G is a pair of points. For each $x \in M$ let $s(x)$ be the point such that x, $s(x)$ is a pair of the collection. Then $T(x) = T(s(x))$. Let $f(x) = \rho(x, s(x))$, where ρ is the metric on M. The

Received November 29, 1939; presented to the American Mathematical Society, October 28, 1939.

[1] This result for the n-to-1 transformation, where n is finite, was noted by O. G. Harrold (see footnote 3), and follows from W. Hurewicz, *Über dimensionserhöhende stetige Abbildungen*, Journal für die reine und angewandte Mathematik, vol. 169(1933), pp. 71–78. An at most $\aleph_0$-to-1 interior transformation cannot increase the dimension. See P. Alexandroff, Comptes Rendus de l'Académie des Sciences de l'URSS, new series, vol. 13(1936), pp. 295–299.

[2] *Interior surface transformations*, this Journal, vol. 4(1938), p. 630.

[3] *The non-existence of a certain type of continuous transformation*, this Journal, vol. 5(1939), pp. 789–793.