THE REPRESENTATION OF FUNCTIONS BY FOURIER INTEGRALS

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1. **Introduction.** Let $f(t)$ be a complex-valued function of the real variable t, bounded in $(-\infty, \infty)$. Cramér [2][1] has recently established necessary and sufficient conditions in order that $f(t)$ admit almost everywhere one of the following representations:

$$f(t) = \int_{-\infty}^{\infty} e^{itx} g(x)\, dx, \tag{g}$$

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dG(x). \tag{G}$$

In these formulas $g(x)$ is a complex-valued function belonging to the class $L(-\infty, \infty)$, and $G(x)$ is a complex-valued function of bounded variation in $(-\infty, \infty)$. Cramér also considered the case in which $G(x)$ is real, bounded and non-decreasing.

In this note we establish necessary and sufficient conditions for the representation almost everywhere of $f(t)$ as a Fourier integral of the above types, $G(x)$ and $g(x)$ belonging to certain other special classes. We obtain also another characteristic condition for the representation of type (g) with $g(x) \in L(-\infty, \infty)$. We also make some applications of the developed method.[2]

2. **The summation function** $s(t)$.[3] Let $s(t)$ be a function which satisfies the following conditions:

$$\int_{-\infty}^{\infty} | s(t) |\, dt < M, \tag{1}$$

$$s(t) = \int_{-\infty}^{\infty} e^{itx} K(x)\, dx, \tag{2}$$

(3) $K(x)$ is real, non-negative, even and $O(| x |^{-1-\alpha})$ as $| x | \to \infty$, α being a positive number; and

$$s(0) = \int_{-\infty}^{\infty} K(x)\, dx = 1. \tag{4}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The author wishes to express his indebtedness to Professor J. D. Tamarkin, who read the manuscript and suggested improvements.

[3] These functions were already considered, before Cramér, by Bochner for a related purpose; see [1], p. 47. Cramér does not assume that $K(x)$ is even and $O(| x |^{-1-\alpha})$ as $| x | \to \infty$; but there is no practical restriction in assuming this condition, which is indeed fulfilled by all the particular functions cited by Cramér as examples (namely, the summation factors of Weierstrass, Poisson and Cesáro).